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Dynamics and optimal control strategies of Corruption model

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Abstract

In this work, we propose a nonlinear mathematics effect model of media on the phenomenon of corruption. We suggest a model that is more general than the ones we are familiar with in this domain, as we work in a structure of nine compartments. This model is important because it explains how corruption spreads in society. Moreover, we have proved the existence and the uniqueness of the solution through the fixed point theorem. The equilibria of the model are determined, and their stability is thoroughly studied. We argue that the corruption-free equilibrium is stable when $\mathcal{R}_0$ is less than one. The endemic equilibrium, which indicates the presence of corruption in the community, exists only when $\mathcal{R}_0 > 1$. Based on the principle of Pontryagin's maximum, an assessment of the requirements for optimal control of corruption spread. We perform extensive numerical simulations to support the analytical results.

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1. Introduction

Many people treat corruption as a routine habit. Corruption can be defined as an aggregate of improper and dishonest acts by those in positions of power such as executives and government officials to increase their profits. The examples of corruption above include giving and accepting bribes and improper gifts, illegal government affairs, fraud or deception, forgery, money transfer, cheating, and money laundering. No country is free of corruption although the level of its acceptance varies considerably from one country to another.

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Corruption is deeply ingrained in society, even to the point that certain countries view it as a social norm. Most countries now have anti-corruption regulations. The often exhausted use of the expression "corruption" helps to stigmatize corruption itself [24]. The money that was supposed to support the minimum level of poverty in developing countries affected by the feds of all kinds was transferred to the benefit of a specific group of people who are considered as the corrupting group. The Robert dictionary defines corruption as an alteration of judgment, taste, language, etc., and as a fact of moral corruption, debasement and perversion. The fight against corruption at the national and international levels remains an issue of great importance with corruption intolerance increasing worldwide. Thus, mathematical modeling applies mathematical principles, namely equations, formulas as well as figures to show what is real. So, mathematical models are relevant because they explain the mathematical core of a particular form
without the need to go into external details. The role of models is to focus on some aspects of the problem with other dimensions being abstracted. Mathematical models are thus relevant because they illustrate the mathematical core of a given context without extraneous information [14].
The relationship between elements of the immune system and cancer is examined and adjusted mathematically by Özdemir and Uçar [1] through the use of the Atangana-Baleanu derivative. The fractional immune system-cancer model's existence and uniqueness are investigated, and numerical simulations are provided using a predictor-corrector technique. While the objective of [2] is to study in depth a model of smoking that is influenced mainly by willpower and educational initiatives through the CF and AB variants. The fixed point approach gets used to prove the existence and uniqueness results for both fractional order models. The objective of [3] is in the first to use Atangana-Baleanu (AB) derivative in conjunction with the fractional SAIDR model, which is reliant on the SAIDR model. The effect of quarantine and isolation in COVID-19 is analyzed in detail by Hamou et al.[9]. The authors of [4] study an e-cigarette smoking model using the Atangana-Baleanu fractional derivative to analyze the phenomenon of smoking. Evirgen et Al. [5] investigated a comprehensive treatment of the newly defined Atangana-Baleanu (AB) fractional derivative in an HIV infection model of $CD4+T$ cells. Uçar [6] take into account a computer virus spreading model that utilizes an Atangana-Baleanu derivative in the sense of Caputo with nonlocal and nonsingular kernels. Uçar and Özdemir [7] discussed the first-ever mathematical model that is presented and looks at how the immune system and cancer cells interact when IL-12 cytokine and an anti-PD-L1 inhibitor are added. Uçar [8] employed the recently implemented Atangana-Baleanu fractional derivative with the Mittag-Leffler kernel to closely study the fundamental SEIRA (susceptible-exposed-infectious-removed-antidotal) model related to computer worms. Hamou et Al. [10] evaluated the effect of multi-order property in fractional derivatives to predict COVID-19. Hamou et Al. [11] reviewed a co-infection model of Hantavirus disease in Belgium.

In the context of the corruption issue and in the planning of intervention programs that aim to control and minimize corruption, mathematical models with optimal control analysis are an ideal step. Corruption has been the subject of relatively few scientific publications as far as our awareness. In [13], Alemneh proposed a nonlinear model of corruption. The basic reproduction number $\mathcal{R}_0$, the point of equilibrium with zero corruption, and the corruption-endemic equilibrium point are all determined. The local and global asymptotic dynamic conditions of the uncorrupted and the endemic equilibrium are committed. The conditions needed for the optimal control of the transmission of corruption are derived using Pontryagin's maximal principle [40, 37]. Abdulrahman [12] developed a model of human behavior in a deterministic corrupt population. They determined the corruption reproduction ratio $\mathcal{R}_0$, the uncorrupted equilibrium point, and the endemic equilibrium point. They concluded that it is difficult to eliminate corruption, but it can be reduced to a level where it can be controlled. In [20], a SIR model for the dynamics of corruption was created by the authors. As a result, they found the number of $\mathcal{R}_0$ and the equilibrium points both corruption-free and endemic, additionally, the model was expanded to enable optimal control utilizing a single optimal control strategy. In [33] Lemecha and Feyissa taking into account the awareness created by anti-corruption and prison counseling as well as the awareness that results from the fight against corruption, Lemecha presented a model to examine the problem of corruption in which they mentioned the need to provide the corrupted individuals in the prison. The authors of [35] propose a novel mathematical model for the dynamics of moral corruption that includes extensive age-appropriate sexual knowledge and therapy

and guidance. Yusuf et Al. [30] have proposed and developed a mathematical model of corruption with a constant recruitment rate, the number of reproduction $\mathcal{R}_0$ is obtained, numerical simulations have been conducted and found that corruption can only be minimized if at least 40% of the corrupt are attacked and denounced by social media accounts.
In this essay, we construct a new mathematical model. We demonstrate the existence and uniqueness of the solutions and then determine the stability associated with the equilibrium point where corruption is absent and the equilibrium point where corruption is persistent. Furthermore, the corruption reproduction number $\mathcal{R}_0$ for the model is computed, and the sensitivity analysis of the model parameters is also studied. We prove the importance of media, sanctions, and publicity in raising awareness through an optimal control model, and the efficacy of our numerical results is presented with the support of figures.

2. Basic preliminaries

The flowing definitions and theorems are used in the next sections of the paper, the existence will be proved using fixed point theorem. The stability of model equilibrium point will be discussed using the flowing theorems

Theorem 2.1 ([44])**.** *If x_0 is an equilibrium point for the differential equation $\dot{x} = f(x)$ and if all eigenvalues of the linear transformation $Df(x_0)$ (Jacobean matrix) have negative real parts, then x_0 is asymptotically stable.*

The above Theorem is used only to prove the local dynamic of the equilibrium point. For the global dynamics we will use the Lyaponov function.

Definition 1 ([44])**.** A continuous function $V : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is an open set with $x_0 \in U$, is called a Lyapunov function for the differential equation at x_0 provided that

(i) $V(x_0) = 0$,

(ii) $V(x) > 0$ for $x \in U - \{x_0\}$,

(iii) the function $x \mapsto \operatorname{grad} V(x)$ is continuous for $x \in U - \{x_0\}$, and, on this set, $\dot{V}(x) := \operatorname{grad} V(x) \cdot f(x) \leq 0$.

If, in addition, (iv) $\dot{V}(x) < 0$ for $x \in U - \{x_0\}$, then V is called a strict Lyapunov function.

Theorem 2.2 (Lyapunov's Stability Theorem [44])**.** *If x_0 is a rest point for the differential equation (1.15) and V is a Lyapunov function for the system at x_0, then x_0 is stable. If, in addition, V is a strict Lyapunov function, then x_0 is asymptotically stable.*

3. Description of the model

In this model, the whole population N is distributed into nine categories of susceptible individuals who are not exposed to social media S_N , sensitive individuals who are in contact with the world of social media S_E corrupt individuals C, imprisoned individuals J, those who are exposed to a corrupt person but do not execute him are exposed individuals E_C , E_{NC} individuals who are not exposed to a corrupt person, semi-recovered individuals R_s (this population cannot be 100% honest), recovered individuals R, and honest individuals H.
Imagine that there is a positive recruitment Π in the susceptible category by the birth, immigration, or implantation of seeds, where, Γ is the proportion of sensitives who are not yet exposed to social media, φ is the rate at which susceptible individuals that are not exposed to social media become exposed to social media. Sensitive individuals S_N shall have a rate of contact with individuals E_{NC} at a rate β, The rate of social media use is defined by f. Individuals in social classes S_N, S_E, C or J become honest due to public

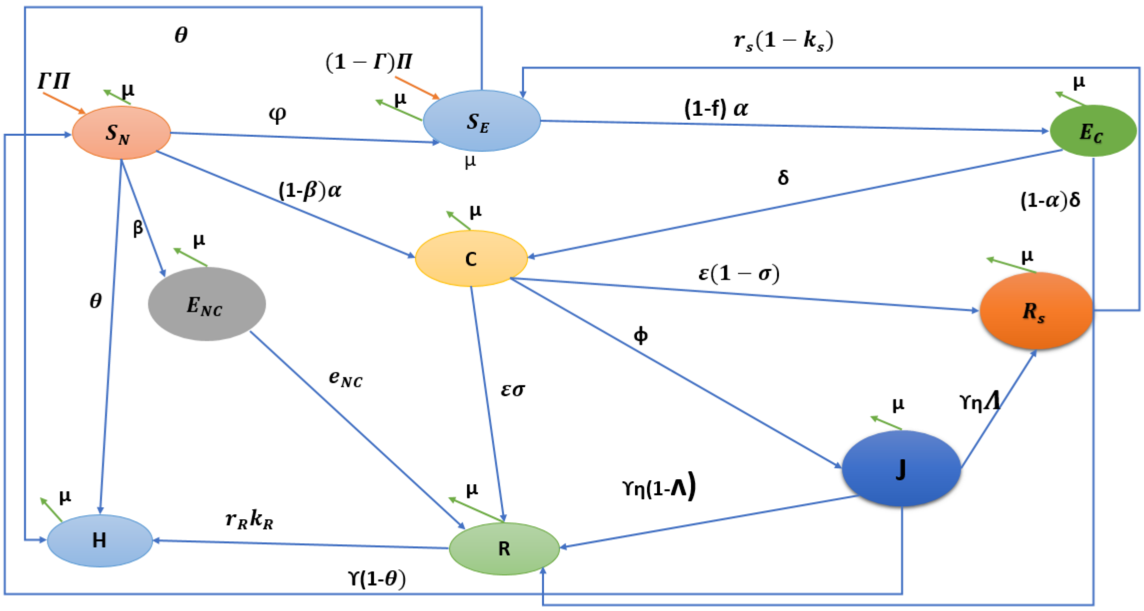


Figure 1: Diagram of the corruption transmission model.

awareness of the danger of corruption on the economy at a rate θ, individuals in E_C then interact with individuals in C and become infected at the rate of δ, ϵ is the proportion of individuals who join R or R_s from C, ϕ is the rate at which people who are corrupt are put in jail σ is the proportion of individuals who join the recovered sub-population from the corrupted population, Λ is the rate at which imprisoned individuals are semi-recovered is the rate at which imprisoned individuals are semi-recovered, η is the proportion of individuals who join R or R_s from J, k_s is the proportion of individuals who join S_E from R_s, r_s is the rate at which semi-recovered individuals are honest, k_R is the proportion of individuals who join H from R and r_R is the rate at which recovered individuals are honest, γ is the rate at which the jailed individuals become susceptible (S_N) to corruption, ω is the decreasing embarrassment rate, e_{NC} is the rate at which E_{NC} individuals are recovered, and become corrupted at the rate of $\alpha = p(1 - e)$, where p is the probability of becoming corrupt when a susceptible individual interact with corrupt individuals, e is the effort rate against corruption.

To always study a mathematical model of a phenomenon, in reality, it is necessary to invoke this relationship by adopting a set of conditions and we list them in the following as indicated in [41]

- People who are expected to be corrupt have the same probability of being corrupt.
- The corrupt individual forces susceptible individuals to engage in corruption when they interact.
- Due to biological significance, the model's parameters are non-negative.
- After being recovered after some time, individuals can become either sensitive or corrupt.
- Corruption's spread in society is similar to the growth of an epidemic disease.
- The rate of recruitment of a new individual into the susceptible and corrupt classes is through birth and immigration.

We used a graph where the vertices are the classes of individuals, and the stops represent the links between the compartments, we assumed that there is a positive recruitment Π. From this class, a subpopulation will join the subpopulation of honest individuals H at rate θ. The individual S_N will have a β contact rate with

E_{NC}. Of these individuals, $(1-\beta)$ moved at an (α) rate to the corrupt compartment, and the rest of the population will move to S_E at the rate φ.
For more details on other classes, see the compartmental diagram in Figure 2, which represents the model of the corruption, and all of the parameter descriptions provided in Table 1.

Therefore, the model can be expressed as an ordinary system of differential equations as shown below:

$$\begin{cases} \dfrac{dS_N}{dt} = \Gamma\Pi + \gamma(1-\theta)J - (\varphi+\beta+\mu+\theta)S_N - \alpha(1-\beta)S_N C, \\ \dfrac{dS_E}{dt} = (1-\Gamma)\Pi + \varphi S_N + r_s(1-k_s)R_s - (1-f)\alpha S_E C - (\theta+\mu)S_E, \\ \dfrac{dE_C}{dt} = (1-f)\alpha S_E C - (\delta+\mu)E_C, \\ \dfrac{dE_{NC}}{dt} = \beta S_N - (e_{NC}+\mu)E_{NC}, \\ \dfrac{dC}{dt} = (1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon+\phi+\mu)C, \\ \dfrac{dJ}{dt} = \phi C - (\gamma\eta + \gamma(1-\theta)+\mu)J, \\ \dfrac{dR_s}{dt} = \gamma\eta\Lambda J + \epsilon(1-\sigma)C - (r_s+\mu)R_s, \\ \dfrac{dR}{dt} = (1-\alpha)\delta E_C + \gamma\eta(1-\Lambda)J + e_{NC}E_{NC} + \epsilon\sigma C - (r_R k_R + \mu)R, \\ \dfrac{dH}{dt} = r_R k_R R + \theta(S_N+S_E) - \mu H. \end{cases} \tag{1}$$

with the initial condition
$S_N(0) = S_{N,0} \geq 0, S_E(0) = S_{E,0} \geq 0, E_C(0) = E_{C,0} \geq 0, E_{NC}(0) = E_{NC,0} \geq 0,$
$C(0) = C_0 \geq 0, J(0) = J_0 \geq 0, R_s(0) = R_{s,0} \geq 0, R(0) = R_0 \geq 0, et\, H(0) = H_0 \geq 0$

Justifying the fact of adding the third, fourth, seventh, and eighth equations in (1). The third equation reflects the situation of individuals who are in contact with corrupt individuals, so there is a probability that they will be corrupted. The fourth equation reflects the situation of individuals who are likely to be corrupt but have not had the opportunity to do so. The seventh and eighth express, in order, a class of semi-recoverable individuals, that is depending on the social environment, they cannot be honest, on the contrary, the individuals of class R can be recoverable with a probability, that is to say, become honest.

4. Qualitative analysis of the model

Let N the total population, then

$$N = S_N + S_E + E_C + E_{NC} + C + J + R_s + R + H.$$

We have

$$\frac{dN}{dt} = \Pi - \mu(S_N(t) + S_E(t) + E_C(t) + E_{NC}(t) + C(t) + J(t) + R_s(t) + R(t) + H(t)) = \Pi - \mu N(t),$$

then

$$\int \frac{dN}{dt} = \int (\Pi - \mu N(t))dt.$$

So $N(t) = N(0)\exp(-\int_0^t \mu ds) + \int_0^t \Pi \exp(-\int_s^t \mu d\vartheta),$

if $\quad N(0) \leq \frac{\Pi}{\mu},$

thus $N(t) \leq \frac{\Pi}{\mu}\exp(-\int_0^t \mu ds) + \int_0^t \Pi \exp(-\mu(t-s)ds) \leq \frac{\Pi}{\mu}\exp(-\int_0^t \mu ds) + \frac{\Pi}{\mu}$,

when $t \to +\infty$, $\quad N(t) \leq \frac{\Pi}{\mu}$
The feasible region for the model (1) is given by

$$\Omega = \left\{ (S_N, S_E, E_C, E_{NC}, C, J, R_s, R, H) \in \mathbb{R}_+^9; \middle| \; S_N, S_E, E_C, E_{NC}, C, J, R_s, R, H \geq 0, N \leq \frac{\Pi}{\mu} \right\}.$$

Clearly, Ω is positively invariant with the system presented by equation (2), in which the model is identified as epidemiologically meaningful and well-posed mathematically. The diagram of the model compartments with parameters are presented in Figure 2.

4.1. Existence and uniqueness of the system solutions

The existence of the solution system is demonstrated by applying the fixed point theorem.
Consider $\mathcal{H} = (C(J))^9$, and C(J) be a Banach field of continuous functions on the interval $J \subset \mathbb{R} \to \mathbb{R}$ with the norm

$$\begin{aligned} &\|(S_N(t), S_E(t), E_C(t), E_{NC}(t), C(t), J(t), R_s(t), R(t), H(t))\| \\ &= \|S_N\|_\infty + \|S_E\|_\infty + \|E_C\|_\infty + \|E_{NC}\|_\infty + \|C\|_\infty + \|J\|_\infty + \|R_s\|_\infty + \|R\|_\infty + \|H\|_\infty, \end{aligned}$$

where $\|.\|_\infty$ indicates the supremum norm in $C(J)$.
For simplicity, we consider

$$\begin{aligned} &\Theta_1(t, S_N) = \Gamma\Pi + \gamma(1-\theta)J - (\varphi + \beta + \mu + \theta)S_N - \alpha(1-\beta)S_N C, \\ &\Theta_2(t, S_E) = (1-\Gamma)\Pi + \varphi S_N + r_s(1-k_s)R_s - (1-f)\beta S_E C - (\theta + \mu)S_E, \\ &\Theta_3(t, E_C) = (1-f)\beta S_E C - (\delta + \mu)E_C, \\ &\Theta_4(t, E_{NC}) = \beta S_N - (e_{NC} + \mu)E_{NC}, \\ &\Theta_5(t, C) = (1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon + \phi + \mu)C, \\ &\Theta_6(t, J) = \phi C - (\gamma\eta + \gamma(1-\theta) + \mu)J, \\ &\Theta_7(t, R_s) = \gamma\eta\Lambda J + \epsilon(1-\sigma)C - (r_s + \mu)R_s, \\ &\Theta_8(t, R) = (1-\alpha)\delta E_C + \gamma\eta(1-\Lambda)J + e_{NC}E_{NC} + \epsilon\sigma C - (r_R k_R + \mu)R, \\ &\Theta_9(t, H) = r_R k_R R + \theta(S_N + S_E) - \mu H. \end{aligned}$$

For proving the existence Theorem, we shall assume that
$\|S_N\| \leq c_1, \|S_E\| \leq c_2, \|E_C\| \leq c_3, \|E_{NC}\| \leq c_4, \|C\| \leq c_5, \|J\| \leq c_6, \|R_s\| \leq c_7, \|R\| \leq c_8, \|H\| \leq c_9$ where $c_i, i = 1, ..., 9$ are positive constants. Thus, we note

$$\begin{aligned} &\kappa_1 = (\varphi + \beta + \mu + \theta) + \alpha(1-\beta)c_4, \\ &\kappa_2 = (1-f)\beta c_5 - (\theta + \mu), \\ &\kappa_3 = (\delta + \mu), \\ &\kappa_4 = (e_{NC} + \mu), \\ &\kappa_5 = (1-\beta)\alpha c_2 + \alpha\delta c_3 - (\epsilon + \phi + \mu), \\ &\kappa_6 = (\gamma\eta + \gamma(1-\theta) + \mu), \\ &\kappa_7 = (r_s + \mu), \\ &\kappa_8 = (r_R k_R + \mu), \\ &\kappa_9 = \mu. \end{aligned}$$

Theorem 4.1. *The kernels, $\Theta_{i=1,...9}$ are valid for the Lipschitz condition and the contraction if the presented inequality holds*

$$0 \leq \kappa_i < 1, \quad for\; i = 1, ..., 9.$$

Proof. Let S_{N_1} and S_{N_2} be two functions, then

$$\begin{aligned}\|\Theta_1(t,S_{N_1})-\Theta_1(t,S_{N_2})\| &= \|-((\varphi+\beta+\mu+\theta)-\alpha(1-\beta)C)(S_{N_1}-S_{N_2})\|,\\ &\leq [(\varphi+\beta+\mu+\theta)+\alpha(1-\beta)\|C\|\|S_{N_1}(t)-S_{N_2}(t)\|,\\ &\leq [(\varphi+\beta+\mu+\theta)+\alpha(1-\beta)c_4]\|S_{N_1}(t)-S_{N_2}(t)\|.\end{aligned}$$

Thus

$$\|\Theta_1(t,S_{N_1})-\Theta_1(t,S_{N_2})\| \leq \kappa_1\|S_{N_1}(t)-S_{N_2}(t)\|.$$

Hence, for Θ_1 the Lipschitz condition is obtained. Likewise, for Θ_2, Θ_3, Θ_4, Θ_5, Θ_6, Θ_7, Θ_8 and Θ_9, the Lipschitz condition can be conveniently verified and is the same as given below

$$\begin{aligned}\|\Theta_2(t,S_{E_1})-\Theta_2(t,S_{E_2})\| &\leq \kappa_2\|S_{E_1}(t)-S_{E_2}(t)\|.\\ \|\Theta_3(t,E_{C_1})-\Theta_3(t,E_{C_2})\| &\leq \kappa_3\|E_{C_1}(t)-E_{C_2}(t)\|.\\ \|\Theta_4(t,E_{NC_1})-\Theta_4(t,E_{NC_2})\| &\leq \kappa_4\|E_{NC_1}(t)-E_{NC_2}(t)\|.\\ \|\Theta_5(t,C_1)-\Theta_5(t,C_2)\| &\leq \kappa_5\|C_1(t)-C_2(t)\|.\\ \|\Theta_6(t,J_1)-\Theta_6(t,J_2)\| &\leq \kappa_6\|J_1(t)-J_2(t)\|.\\ \|\Theta_7(t,R_{s_1})-\Theta_7(t,R_{s_2})\| &\leq \kappa_7\|R_{s_1}(t)-R_{s_2}(t)\|.\\ \|\Theta_8(t,R_1)-\Theta_8(t,R_2)\| &\leq \kappa_8\|R_1(t)-R_2(t)\|.\\ \|\Theta_9(t,H_1)-\Theta_9(t,H_2)\| &\leq \kappa_9\|H_1(t)-H_2(t)\|.\end{aligned}$$

□

Since model (1) follows the population of humans, all its variables of state and the relative parameters must be positive for the coming time. This shall be demonstrated by the theorem below:

Theorem 4.2. *All solutions $(S_N, S_E, E_N, E_{NC}, C, J, R_s, R, H)$ are positive whenever $t \geq 0$.*

Proof. Count on the state variables all being continuous. So, it is simple to determine from the system (1) that:
$\frac{dS_N}{dt} \geq (\varphi+\beta+\mu+\theta)S_N - \alpha(1-\beta)S_N C.$
$\frac{dS_E}{dt} \geq -(1-f)\alpha S_E C - (\theta+\mu)S_E.$
$\frac{dE_C}{dt} \geq -(\delta+\mu)E_C.$
$\frac{dE_{NC}}{dt} \geq -(e_{NC}+\mu)E_{NC}.$
$\frac{dC}{dt} \geq -(\epsilon+\phi+\mu)C.$
$\frac{dJ}{dt} \geq -(\gamma\eta+\gamma(1-\theta)+\mu)J.$
$\frac{dR_s}{dt} \geq -(r_s+\mu)R_s.$
$\frac{dR}{dt} \geq -(r_R k_R+\mu)R.$
$\frac{dH}{dt} \geq -\mu H.$
Then, using the variation of constant formula:
$\frac{dS_N}{dt} \geq S_N(0)\exp(-(\varphi+\beta+\mu+\theta+\alpha(1-\beta)C)t) \geq 0.$
$\frac{dS_E}{dt} \geq S_E(0)\exp(-((1-f)\alpha C+(\theta+\mu))t) \geq 0.$
$\frac{dE_C}{dt} \geq E_C(0)\exp(-(\delta+\mu)t) \geq 0.$
$\frac{dE_{NC}}{dt} \geq E_N C(0)\exp(-(e_{NC}+\mu)t) \geq 0.$
$\frac{dC}{dt} \geq C_0(0)\exp(-(\epsilon+\phi+\mu)t) \geq 0.$
$\frac{dJ}{dt} \geq J_0(0)\exp(-(\gamma\eta+\gamma(1-\theta)+\mu)t) \geq 0.$
$\frac{dR_s}{dt} \geq R_s(0)\exp(-(r_s+\mu)t) \geq 0.$

Table 1: Parameters of the model

Parameters	Description
$S_N(t)$	Quantity of people who are likely to be out of contact with social media.
$S_E(t)$	Quantity of susceptible individuals that are in contact with social media.
$E_C(t)$	Quantity of individuals who are exposed to C but do not execute them.
$E_{NC}(t)$	Quantity of individuals who are not exposed to a corrupt person.
$C(t)$	Quantity of corrupt individuals.
$J(t)$	Quantity of jailed individuals.
$R_s(t)$	Quantity of semi-recovered individuals.
$R(t)$	Quantity of recovered individuals.
$H(t)$	Quantity of honest individuals.
Γ	Proportion of recruited individuals who are not exposed to social media.
μ	Natural death rate.
Π	Recruitment number.
β	The interaction of S_E and E_{NC}.
θ	Rate at which individuals in the compartments S_N, S_E, C or J become honest.
k_R	Proportion of individuals who join H from R.
ω	Decreasing rate of embarrassment
r_R	Rate at which recovered individuals are Honest.
e_{NC}	Rate at which E_NC individuals are recovered.
Λ	Rate at which imprisoned individuals are semi-recovered.
f	Social media usage rate
k_s	Proportion of individuals who join S_E from R_s.
r_s	Rate at which semi-recovered individuals are honest.
φ	Rate at which S_N become exposed to social media.
ϕ	Rate at which Corrupted individuals are in jail.
ϵ	Proportion of individuals who join R or R_s from C.
δ	Rate at which E_C individuals are corrupted.
η	proportion of individuals who join R or R_s from J.

$\frac{dR}{dt} \geq R_0(0)\exp(-(r_R k_R + \mu)t) \geq 0.$
$\frac{dH}{dt} \geq H_0(0)\exp(-\mu t) \geq 0.$
Consequently, the entire solutions sets are positive for $t \geq 0$.

□

4.2. Local dynamic of the corruption free-equilibrium

Because $R(t), H(t)$, and $E_{NC}(t)$ re not present in the first three equations and the fifth, sixth, and seventh, the system (2) can be expressed as below

$$\begin{cases} \dfrac{dS_N}{dt} = \Gamma\Pi + \gamma(1-\theta)J - (\varphi+\beta+\mu+\theta)S_N - \alpha(1-\beta)S_N C, \\ \dfrac{dS_E}{dt} = (1-\Gamma)\Pi + \varphi S_N + r_s(1-k_s)R_s - (1-f)\alpha S_E C - (\theta+\mu)S_E, \\ \dfrac{dE_C}{dt} = (1-f)\alpha S_E C - (\delta+\mu)E_C, \\ \dfrac{dC}{dt} = (1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon+\phi+\mu)C, \\ \dfrac{dJ}{dt} = \phi C - (\gamma\eta + \gamma(1-\theta)+\mu)J, \\ \dfrac{dR_s}{dt} = \gamma\eta\Lambda J + \epsilon(1-\sigma)C - (r_s+\mu)R_s. \end{cases} \tag{2}$$

The uncorrupted equilibrium point of the model is achieved by taking all the equations of the model (2) to zero and allowing. $E_C = 0, C = 0, J = 0$ and $R_s = 0$.

We then obtain,

$$\mathcal{E}_0 = (S_N^0, S_E^0, 0, 0, 0, 0),$$

where

$$S_N^0 = \frac{\Gamma\Pi}{(\varphi+\beta+\mu+\theta)} \quad , \quad S_E^0 = \frac{(1-\Gamma)\Pi + \varphi S_N^0}{\theta+\mu} = \frac{\Pi(\varphi + (\beta+\mu+\theta)(1-\Gamma))}{(\theta+\mu)(\varphi+\beta+\mu+\theta)}.$$

4.3. The corruption reproduction number $\mathcal{R}_0$

In epidemiology, the basic reproduction number, or basic reproductive number of an infection is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. The definition assumes that no other individuals are infected or immunized (naturally or through vaccination). The most important uses of the basic reproduction number denoted by $\mathcal{R}_0$ are determining if an emerging infectious disease can spread in a population and determining what proportion of the population should be immunized through vaccination to eradicate a disease. The corruption reproduction number of the model is important for analyzing the stability of the equilibrium points. Furthermore, $\mathcal{R}_0$ is used to estimate the anticipated number of secondary connections arising from the introduction of a newly detected individual among a sensitive community. Based on the generation matrix method as detailed in [36], we obtain $\mathcal{R}_0$ directly from the model. As a first step to get $\mathcal{R}_0$, we rewrite the equations of the model beginning with the recently infected categories

$$\begin{cases} \dfrac{dE_C}{dt} = (1-f)\alpha S_E C - (\delta+\mu)E_C, \\ \dfrac{dE_{NC}}{dt} = \beta S_N - (e_{NC}+\mu)E_{NC}, \\ \dfrac{dC}{dt} = (1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon+\phi+\mu)C, \\ \dfrac{dJ}{dt} = \phi C - (\gamma\eta + \gamma(1-\theta)+\mu)J, \\ \dfrac{dR_s}{dt} = \gamma\eta\Lambda J + \epsilon(1-\sigma)C - (r_s+\mu)R_s. \end{cases}$$

The basic reproduction number $\mathcal{R}_0$ is measured as the radius of the spectral ρ of the generation matrix FV^{-1}.

The matrices of F and V are the result of the infected classes (i.e. E_C, E_{NC}, C, J and R_s) at the equilibrium point $\mathcal{E}_0$, and so we have:

$$\mathbb{F} = \begin{pmatrix} (1-f)\alpha S_E C \\ \beta S_N \\ (1-\beta)\alpha S_N C \\ 0 \\ 0 \end{pmatrix},$$

and

$$\mathbb{V} = \begin{pmatrix} (\delta+\mu)E_C \\ e_{NC}+\mu)E_{NC} \\ \epsilon+\phi+\mu)C-\alpha\delta E_C \\ \gamma\eta+\gamma(1-\theta)+\mu)J-\phi C \\ (r_s+\mu)R_s-\varepsilon(1-\sigma)C-\gamma\eta\Lambda J \end{pmatrix},$$

as $F=[\frac{\partial \mathbb{F}}{\partial X_j}]$ and $V=[\frac{\partial \mathbb{V}}{\partial X_j}]$, where $X_j=(E_C,E_{NC},C,J,R_s)$.
We have

$$F=\frac{\partial \mathbb{F}}{\partial X_j}(\mathcal{E}_0) = \begin{pmatrix} 0 & 0 & (1-f)\alpha S_E^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\beta)\alpha S_N^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V=\frac{\partial \mathbb{V}}{\partial X_j}(\mathcal{E}_0) = \begin{pmatrix} \delta+\mu & 0 & 0 & 0 & 0 \\ 0 & e_{NC}+\mu & 0 & 0 & 0 \\ -\alpha\delta & 0 & (\varepsilon+\phi+\mu) & 0 & 0 \\ 0 & 0 & -\phi & \gamma\eta+\gamma(1-\theta)+\mu) & 0 \\ 0 & 0 & -\epsilon(1-\sigma) & -\gamma\eta\Lambda & r_s+\mu \end{pmatrix}.$$

We can see that
$|V|=(r_s+\mu)(\gamma\eta+\gamma(1-\theta)+\mu)(\varepsilon+\phi+\mu)(\delta+\mu)(e_{NC}+\mu)$.

Then

$$com(V) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{pmatrix},$$

with
$c_{11}=(r_s+\mu)(\gamma\eta+\gamma(1-\theta)+\mu)(e_{NC}+\mu)(\varepsilon+\phi+\mu)$,
$c_{12}=c_{21}=c_{23}=c_{24}=c_{25}=c_{31}=c_{32}=c_{41}=c_{42}=c_{43}=0$,
$c_{13}=(r_s+\mu)(\gamma\eta+\gamma(1-\theta)+\mu)\alpha\delta(e_{NC}+\mu)$,
$c_{14}=-(r_s+\mu)(e_{NC}+\mu)\alpha\delta\phi$,
$c_{15}=-(e_{NC}+\mu)\alpha\delta(\phi\gamma\eta\Lambda+(\gamma\eta+\gamma(1-\theta)+\mu)\varepsilon(1-\sigma))$,
$c_{22}=(r_s+\mu)(\gamma\eta+\gamma(1-\theta)+\mu)(\delta+\mu)(\varepsilon+\phi+\mu)$,
$c_{33}=(r_s+\mu)(\gamma\eta+\gamma(1-\theta)+\mu)(\delta+\mu)(e_{NC}+\mu)$,
$c_{34}=(r_s+\mu)\phi(\delta+\mu)(e_{NC}+\mu)$,
$c_{35}=(\delta+\mu)(e_{NC}+\mu)(\phi\gamma\Lambda+(\gamma\eta+\gamma(1-\theta)+\mu)\varepsilon(1-\sigma))$,
$c_{44}=(r_s+\mu)(\varepsilon+\phi+\mu)(\delta+\mu)(e_{NC}+\mu)$,
$c_{45}=\gamma\eta\Lambda(\varepsilon+\phi+\mu)(\delta+\mu)(e_{NC}+\mu)$,
$c_{51}=c_{52}=c_{53}=c_{54}=0$,

$c_{55} = (\gamma\eta + \gamma(1-\theta) + \mu)(\varepsilon + \phi + \mu)(\delta + \mu)(e_{NC} + \mu)$,
then

$$V^{-1} = \frac{1}{|V|}\begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} & c_{51} \\ c_{12} & c_{22} & c_{32} & c_{42} & c_{52} \\ c_{13} & c_{23} & c_{33} & c_{43} & c_{53} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{54} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} \end{pmatrix},$$

thus

$$FV^{-1} = \frac{1}{|V|}\begin{pmatrix} (1-f)\alpha S_E^0 c_{13} & 0 & (1-f)\alpha S_E^0 c_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (1-\beta)\alpha S_N^0 c_{13} & 0 & c_{33}(1-\beta)\alpha S_N^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the effective reproduction number $\mathcal{R}_0$ is given as follows

$$\begin{aligned} \mathcal{R}_0 &= \rho(FV^{-1}) \\ &= \frac{(1-f)S_E^0\alpha^2\delta + (1-\beta)\alpha(\delta+\mu)S_N^0}{(\epsilon+\phi+\mu)(\delta+\mu)} \\ &= \frac{(1-f)\alpha^2\delta((1-\Gamma)\Pi(\varphi+\beta+\mu+\theta) + \varphi\Gamma\Pi) + (1-\beta)\alpha(\delta+\mu)\Gamma\Pi(\theta+\mu)}{(\epsilon+\phi+\mu)(\delta+\mu)(\varphi+\beta+\mu+\theta)(\theta+\mu)} \end{aligned}$$

Theorem 4.3. *The equilibrium $\mathcal{E}_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. However, to demonstrate this theorem, let us first obtain the Jacobian matrix of the system (2)

$$J = \begin{pmatrix} J_{11} & 0 & 0 & J_{14} & \gamma(1-\theta) & 0 \\ \varphi & J_{22} & 0 & J_{24} & 0 & J_{25} \\ 0 & J_{32} & J_{33} & J_{34} & 0 & 0 \\ (1-\beta)\alpha C & 0 & \alpha\delta & J_{44} & 0 & 0 \\ 0 & 0 & 0 & \phi & J_{55} & 0 \\ 0 & 0 & 0 & \epsilon(1-\sigma) & \gamma\eta\Lambda & -(r_s+\mu) \end{pmatrix}. \tag{3}$$

With
$J_{11} = -(\alpha(1-\beta)C + (\varphi+\beta+\mu+\theta))$,
$J_{14} = -\alpha(1-\beta)S_N$,
$J_{22} = -((1-f)\alpha C + (\theta+\mu))$,
$J_{24} = -(1-f)\alpha S_E$,
$J_{25} = r_s(1-k_s)$,
$J_{32} = (1-f)\alpha C$,
$J_{33} = -(\delta+\mu)$,
$J_{34} = (1-f)\alpha S_E$,
$J_{44} = (1-\beta)\alpha S_N - (\epsilon+\phi+\mu)$,
$J_{55} = -(\gamma\eta + \gamma(1-\theta) + \mu)$.

Evaluating the Jacobean matrix (3) at the equilibrium $\mathcal{E}_0$, we obtain

$$J(\mathcal{E}_0) = \begin{pmatrix} -(\varphi+\beta+\mu+\theta) & 0 & 0 & J_{14} & \gamma(1-\theta) & 0 \\ \varphi & -(\theta+\mu) & 0 & J_{24} & 0 & J_{25} \\ 0 & 0 & J_{33} & J_{34} & 0 & 0 \\ 0 & 0 & \alpha\delta & J_{44} & 0 & 0 \\ 0 & 0 & 0 & \phi & J_{55} & 0 \\ 0 & 0 & 0 & \epsilon(1-\sigma) & \gamma\eta\Lambda & -(r_s+\mu) \end{pmatrix}.$$

Based on the Jacobian matrix, a characteristic polynomial was attained in the following form

$$\mathcal{P}(\lambda) = (\varphi + \mu + \beta + \theta + \lambda)(\theta + \mu + \lambda)(r_s + \mu + \lambda)(\gamma\eta + \gamma(1-\theta) + \mu)\bar{\mathcal{P}}(\lambda),$$

then

$$\begin{aligned}
\lambda_1 &= -(r_s + \mu) < 0, \\
\lambda_2 &= -(e_{NC} + \mu) < 0, \\
\lambda_3 &= -(\theta + \mu) < 0, \\
\lambda_4 &= -(\varphi + \beta + \mu + \theta) < 0,
\end{aligned}$$

λ_5 and λ_6 are the roots of

$$\begin{aligned}
\bar{\mathcal{P}}(\lambda) &= \lambda^2 + \lambda((\delta+\mu) - (1-\beta)\alpha S_N^0 + (\epsilon + \phi + \mu)) - [(\delta+\mu)((1-\beta)\alpha S_N^0 - (\epsilon+\phi+\mu)) + (1-f)\alpha S_E^0 \alpha\delta], \\
&= \lambda^2 + \lambda\zeta - \upsilon,
\end{aligned}$$

where
$\zeta = ((\delta+\mu) - (1-\beta)\alpha S_N^0 + (\epsilon+\phi+\mu))$, and $\upsilon = [(\delta+\mu)((1-\beta)\alpha S_N^0 - (\epsilon+\phi+\mu)) + (1-f)\alpha S_E^0 \alpha\delta]$,
we got

$$\begin{aligned}
\Delta &= \zeta^2 + 4\upsilon \\
&= ((\delta+\mu) - (1-\beta)\alpha S_N^0 + (\epsilon+\phi+\mu))^2 + 4[(\delta+\mu)((1-\beta)\alpha S_N^0 - (\epsilon+\phi+\mu)) + (1-f)\alpha S_E^0\alpha\delta] \\
&= ((\delta+\mu) - (\epsilon+\phi+\mu))^2 + ((1-\beta)\alpha S_N^0 - (\epsilon+\phi+\mu))^2 + 2(\delta+\mu)(1-\beta)\alpha S_N^0 \\
&\quad + 4(1-f)\alpha S_E^0\alpha\delta - (\epsilon+\phi+\mu)^2,
\end{aligned}$$

it is simple to observe that $2(\delta+\mu)(1-\beta)\alpha S_N^0 + 4(1-f)\alpha S_E^0\alpha\delta > (\epsilon+\phi+\mu)^2$,
then we conclude that $\Delta > 0$,
and thus

$$\begin{aligned}
\lambda_5 &= -\frac{\zeta + \sqrt{\Delta}}{2} < 0, \\
\lambda_6 &= \frac{-\zeta + \sqrt{\Delta}}{2}
\end{aligned}$$

$\lambda_6 < 0$ for $\Delta < \zeta^2$,
then $\mathcal{R}_0(\epsilon+\phi+\mu)(\delta+\mu) < (\epsilon+\phi+\mu)(\delta+\mu)$, means $\mathcal{R}_0 < 1$
while on the other hand we have
$\mathcal{R}_0 < 1$ for $\frac{(1-f)S_E^0\alpha^2\delta + (1-\beta)\alpha(\delta+\mu)S_N^0}{(\epsilon+\phi+\mu)(\delta+\mu)} < 1$,
then $4\upsilon < 0$ includes $4c + \zeta^2 < \zeta^2$, what gives $\lambda_6 < 0$

Therefore, after using the stability Theorem 2.1 , $\mathcal{E}_0$ is locally asymptotically stable.

So that corruption can be removed to some extent if the starting population size of the corrupted members is in the bottom set of the point $\mathcal{E}_0$. □

4.4. Global dynamic of the corruption free-equilibrium

Theorem 4.4. *The corruption free-equilibrium $\mathcal{E}_0$ of the model (2) is globally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. We shall consider the Lyapunov function as follows

$$V = \chi_1 E_C + \chi_2 C \tag{4}$$

where χ_1 and χ_2 are two positive values.
If we differentiate equation (4) concerning t, we discover that.

$$\frac{dV}{dt} = \chi_1 \frac{dE_C}{dt} + \chi_2 \frac{dC}{dt}$$

By replacing dE_C/dt et dC/dt of model (2), we obtain:

$\frac{dV}{dt} = \chi_1((1-f)\alpha S_E C - (\delta+\mu)E_C) + \chi_2((1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon+\phi+\mu)C)$

$\frac{dV}{dt} = [\chi_1(1-f)\alpha S_E + \chi_2(1-\beta)\alpha S_N - \chi_2(\epsilon+\phi+\mu)]C - (\chi_1(\delta+\mu) - \chi_2\alpha\delta)E_C$

In this case, we take $\chi_2 = \frac{(\delta+\mu)}{\alpha\delta}\chi_1$,
then

$$\begin{aligned}\frac{dV}{dt} &= [\chi_1(1-f)\alpha S_E + \chi_2(1-\beta)\alpha S_N - \chi_2(\epsilon+\phi+\mu)]C \\ &= \chi_1 C[(1-f)\alpha S_E + \frac{\delta+\mu}{\alpha\delta}(1-\beta)\alpha S_N - \frac{(\epsilon+\phi+\mu)(\delta+\mu)}{\alpha\delta}],\end{aligned}$$

As $(\epsilon+\phi+\mu)(\delta+\mu) = \frac{(1-f)S_E^0\alpha^2\delta + (1-\beta)\alpha(\delta+\mu)S_N^0}{\mathcal{R}_0}$
thus

$$\begin{aligned}\frac{dV}{dt} &= \chi_1 C[(1-f)\alpha S_E + \frac{\delta+\mu}{\alpha\delta}(1-\beta)\alpha S_N - \frac{(1-f)S_E^0\alpha^2\delta + (1-\beta)\alpha(\delta+\mu)S_N^0}{\mathcal{R}_0\alpha\delta}] \\ &= \chi_1 C[(1-f)\alpha(S_E - \frac{S_E^0}{\mathcal{R}_0}) + \frac{(\delta+\mu)}{\delta}(1-\beta)(S_N - \frac{S_N^0}{\mathcal{R}_0})] \\ &< \chi_1 C[(1-f)\alpha(S_E^0 - \frac{S_E^0}{\mathcal{R}_0}) + \frac{(\delta+\mu)}{\delta}(1-\beta)(S_N^0 - \frac{S_N^0}{\mathcal{R}_0})] \\ &< \chi_1 C[(1-\frac{1}{\mathcal{R}_0})(S_E^0(1-f)\alpha + \frac{(\delta+\mu)}{\delta}(1-\beta)S_N^0)]\end{aligned}$$

well then, from $C < C^0$ and $\frac{dV}{dt} \leq 0$ for $\mathcal{R}_0 < 1$ and $\frac{dV}{dt} = 0$ if and only if $C = 0$.
Hence, according to Theorem 2.2, $\mathcal{E}_0$ becomes globally asymptotically stable. □

4.5. The endemic equilibrium "EE"

There is a unique endemic equilibrium of the system (2) given by

$$EE = (S_N^*, S_E^*, E_C^*, E_{NC}^*, C^*, J^*, R_s^*, R^*, H^*),$$

from where EE is the solution to the steady-state with corruption in the community. It can be achieved by nullifying each equation in (2)

$$\frac{dS_N}{dt} = \frac{dS_E}{dt} = \frac{dE_C}{dt} = \frac{dE_{NC}}{dt} = \frac{dC}{dt} = \frac{dJ}{dt} = \frac{dR_s}{dt} = \frac{dR}{dt} = \frac{dH}{dt} = 0.$$

Then, we obtain

$$S_N^* = \frac{\epsilon+\phi+\mu - \frac{\alpha\delta(1-f)\alpha S_E^*}{\delta+\mu}}{(1-\beta)\alpha} = a_1 S_E^* + b, \quad S_E^* = \frac{-m_3 - m_6 C^*}{m_4 - m_6}. \tag{5}$$

$$E_C^* = \frac{(1-f)\alpha S_E^* C^*}{\delta+\mu} = \frac{-m_5(m_3 + m_6 C^*)}{(\delta+\mu)(m_4 - m_5)}, \quad J^* = \frac{\phi C^*}{\gamma\eta + \gamma(1-\theta) + \mu} = m_1 C^*. \tag{6}$$

$$R_s^* = \frac{\gamma\eta\Lambda m_1 + \epsilon(1-\sigma)}{r_s + \mu} C^* = m_2 C^*. \tag{7}$$

All the expressions are in terms of the C^*
in which C^* is the positive solution of the equation given below:

$$AC^2 + BC + D = 0, \tag{8}$$

with

$$
\begin{aligned}
A &= A_3 m_6 - A_1 m_5,\\
B &= A_1 m_4 - A_0 m_5 + A_2 m_6 + A_3 m_3,\\
D &= m_4 A_0 + A_2 m_3,\\
b &= \frac{\epsilon + \theta + \mu}{(1-\beta)\alpha},\\
a_1 &= \frac{-\alpha\delta(1-f)}{(\delta+\mu)(1-\beta)},\\
A_0 &= \Gamma\Pi + (\varphi + \beta + \mu + \theta)b,\\
A_1 &= \gamma(1-\theta)m_1 - \alpha(1-\beta)b,\\
A_2 &= (\varphi + \beta + \mu + \theta)a_1,\\
A_3 &= \alpha(1-\beta)a_1,\\
m_1 &= \frac{\phi}{\gamma\eta + \gamma(1-\theta) + \mu},\\
m_2 &= \frac{\gamma\eta\Lambda m_1 + \epsilon(1-\sigma)}{r_s + \mu},\\
m_3 &= (1-\Gamma)\Pi + \varphi b,\\
m_4 &= \varphi a_1 - (\theta + \mu),\\
m_5 &= (1-f)\alpha,\\
m_6 &= r_s(1-k_s)m_2.
\end{aligned}
$$

It is necessary to demonstrate that the solution of (8) is both realistic and positive. Consider that

$$A < 0 \ \ and \ \ D > 0 \iff \mathcal{R}_0 > 1.$$

The discriminant of (8) is Δ, such that $\Delta = B^2 - 4AD$. It is simple to observe that $D > 0$, and mention that all model parameters are non-negative. Hence if $R > 1$, then $A < 0$ and $D > 0$.
The results for the endemic equilibrium point are as follows:

Theorem 4.5. *There is a unique endemic stable point for the system (2) whenever $\mathcal{R}_0 > 1$, which is indicated by*
$EE = (S_N^*, S_E^*, E_C^*, E_{NC}^*, C^*, J^*, R_s^*, R^*, H^*)$,
where the expressions of $S_N^*, S_E^*, E_C^*, E_{NC}^*, J^*, R_s^*, R^*$ and H^* are given in (5), (6) and (7) and
$C^* = \frac{-B-\sqrt{B^2-4AD}}{2A}$.

4.6. Local dynamic of endemic equilibrium point "EE"

Theorem 4.6. *The point EE is locally asymptotically stable, where $\mathcal{R}_0 > 1$.*

Proof. To establish the above theorem, we must first derive the Jacobian matrix of the system (2) :

$$
J = \begin{pmatrix}
J_{11} & 0 & 0 & J_{14} & \gamma(1-\theta) & 0\\
\varphi & J_{22} & 0 & J_{24} & 0 & J_{25}\\
0 & J_{32} & J_{33} & J_{34} & 0 & 0\\
(1-\beta)\alpha C & 0 & \alpha\delta & J_{44} & 0 & 0\\
0 & 0 & 0 & \varphi & J_{55} & 0\\
0 & 0 & 0 & \epsilon(1-\sigma) & \gamma\eta\Lambda & -(r_s+\mu)
\end{pmatrix}.
$$

The characteristic polynomial of equation (3) at the endemic point EE is given by

$$\mathcal{P}(\lambda) = \Psi_0 + \lambda\Psi_1 + \lambda^2\Psi_2 + \lambda^3\Psi_3 + \lambda^4\Psi_4 + \lambda^5\Psi_5 + \lambda^6.$$

Where

$\Psi_0 = -J_{110}b_0 + \varphi J_{32}J_{660}(\gamma(1-\theta)\alpha\delta\phi + J_{550}\alpha\delta J_{14}) + a_5 J_{220}J_{660}J_{330}$,
$J_{110} = -J_{11}$,
$J_{220} = -J_{22}$,
$J_{330} = -J_{33}$,
$J_{550} = -J_{55}$,
$J_{660} = -J_{66} = (r_s + \mu)$,

$$\begin{aligned}\Psi_1 =& -(J_{110}b_1 + b_0) + \varphi J_{32}(J_{660}\alpha\delta J_{14} + \gamma(1-\theta)\alpha\delta\varphi + J_{550}\alpha\delta J_{14})\\ &+a_5((J_{220} + J_{660})J_{330} + J_{220}J_{660}),\\ \Psi_2 =& -(J_{110}b_2 + b_1) + \varphi J_{32}\alpha\delta J_{14} + a_5(J_{330} + J_{220} + J_{660}),\end{aligned}$$

$\Psi_3 = -(J_{110}b_3 + b_2) + a_5$,

$$\Psi_4 = -(J_{110}b_4 + b_3),$$

$\Psi_5 = J_{110} - b_4$,
$a_0 = J_{220}J_{660}J_{550}$,
$a_1 = (J_{220} + J_{660})J_{550} + J_{220}J_{660}$,
$a_2 = J_{220} + J_{660} + J_{550}$,
$a_3 = J_{330}J_{440} + \alpha\delta J_{34}$,
$a_4 = J_{440} - J_{330}$,
$a_5 = (1-\beta)\alpha C^*(J_{14}J_{25} - \phi\gamma(1-\theta))$,
$b_0 = a_0a_3 + J_{25}\phi\gamma\eta\Lambda + J_{32}\alpha\delta J_{550}J_{660}$,
$b_1 = a_1a_3 + a_0a_4 + J_{32}\alpha\delta J_{25}(J_{550} + J_{660}) + \epsilon(1-\sigma)J_{25}$,
$b_2 = a_2a_3 + a_1a_4 - a_0 + J_{32}\alpha\delta J_{24}$,
$b_3 = a_3 + a_2a_4 - a_1$,
$b_4 = a_4 - a_2$.

$$\begin{array}{c||cccc}
\lambda^6 & \psi_6 & \psi_4 & \psi_2 & \psi_0\\
\lambda^5 & \psi_5 & \psi_3 & \psi_1 & 0\\
\lambda^4 & s_1 & s_2 & \psi_0 & 0\\
\lambda^3 & s_3 & s_4 & 0 & 0\\
\lambda^2 & s_5 & s_6 & 0 & 0\\
\lambda & s_7 & 0 & 0 & 0
\end{array}$$

$$s_1 = \frac{\psi_5\psi_4 - \psi_3}{\psi_5}, \quad s_2 = \frac{\psi_5\psi_2 - \psi_1}{\psi_5}, \quad s_3 = \frac{s_1\psi_3 - \psi_5 s_2}{s_1},$$

$$s_4 = \frac{s_1\psi_1 - \psi_5\psi_0}{s_1}, \quad s_5 = \frac{s_3 s_2 - s_1 s_4}{s_3}, \quad s_6 = \frac{s_3\psi_0 - s_1 s_4}{s_3}.$$

According to the Routh-Hurwitz criterion [22], the characteristic polynomial has all roots with a non-negative real part if and only if $s_1 > 0$.
So for s_1 to be positive, we must have $\alpha(1-\beta)(S_N - C^*) > \tau_1 + \tau_2$,
As a simplification, we thus obtain $\mathcal{R}_0 > (\frac{\tau_1+\tau_2}{(S_N-C^*)} + (1-f)\alpha^2\delta S_E^0)\frac{\alpha S_N^0}{\epsilon+\phi+\mu}$.
Thus $s_1 > 0$ if $\mathcal{R}_0 > 1$.

Where
$\tau_1 = \frac{J_{11}(a_3+a_2a_4-a_1-b_2}{J_{11}(a_4-a_2)+a_1-a_3-a_2a_4}$.
$\tau_2 = \theta + \mu + B$.
$B = (\varphi + \beta + \mu + \epsilon + \phi + \mu) + (1-f)\alpha C^* + (r_s + \mu) + (\gamma\eta + \gamma(1-\theta) + \mu)$.
Hence, by Theorem 2.1 the endemic equilibrium EE is locally asymptotically stable.

□

4.7. The Global dynamic of endemic equilibrium point "EE"

Theorem 4.7. *When $\mathcal{R}_0 > 1$, the sole endemic steady state of (2) is globally asymptotically stable.*

Proof. For example, it is possible to take the following Lyapunov function:

$$L(S_N, C) = \chi_1(S_N - S_N^* \ln \frac{S_N}{S_N^*}) + \chi_2(C - C^* \ln \frac{C}{C^*}). \tag{9}$$

where $\chi_1 > 0$ and $\chi_2 > 0$ are positive constants to be selected next. Obviously L is C^1, $L(EE) = 0$, and L is strictly positive at other points.
Deriving equation with respect to t, we have

$$\frac{dL(S_N, C)}{dt} = \chi_1(1 - \frac{S_N^*}{S_N})\frac{dS_N}{dt} + \chi_2(1 - \frac{C^*}{C})\frac{dC}{dt}. \tag{10}$$

By substituting $\frac{dS_N}{dt}$ et $\frac{dC}{dt}$ of model (2), we obtain,

$$\begin{aligned}\frac{dL}{dt} =& \chi_1(1 - \frac{S_N^*}{S_N})(\Gamma\Pi + \gamma(1-\theta)J - (\varphi + \beta + \mu + \theta)S_N - \alpha(1-\beta)S_N C) \\ &+ \chi_2(1 - \frac{C^*}{C})((1-\beta)\alpha S_N C + \alpha\delta E_C - (\epsilon + \phi + \mu)C).\end{aligned}$$

From system of equations (2), we have
$\Gamma\Pi + \gamma(1-\theta)J = (a + \alpha(1-\beta)C)S_N^*$ and $\alpha\delta E_C = (b - (1-\beta)\alpha S_N)C^*$. Then

$$\frac{dL}{dt} = -\frac{\chi_1}{S_N}(a + \alpha(1-\beta)C)(S_N - S_N^*)^2 - \frac{\chi_2}{C}(b - (1-\beta)\alpha S_N)(C - C^*)^2. \tag{11}$$

Factorizing equation, we have

$$\frac{dL}{dt} = \frac{(1-\beta)\alpha}{S_N C}(\chi_2 S_N^2(C - C^*)^2 - \chi_1 C^2(S_N - S_N^*)^2) - \frac{1}{S_N C}(\chi_1 C(S_N - S_N^*)^2 a + \chi_2 S_N(C - C^*)^2 b).$$

Where,
$a = (\varphi + \beta + \mu + \theta)$, and $b = (\epsilon + \phi + \mu)$
$\chi_2 S_N(C - C^*)^2 b$ is a positive term, then

$$\begin{aligned}\frac{dL}{dt} &< \frac{(1-\beta)\alpha}{S_N C}(\chi_2 S_N^2(C - C^*)^2 - \chi_1 C^2(S_N - S_N^*)^2) - \frac{1}{S_N C}(\chi_1 C(S_N - S_N^*)^2 a) \\ &< -\frac{\mathcal{R}_0 b \chi_1 C^2(S_N - S_N^*)^2}{S_N C S_N^0} + \frac{\chi_1(1-f)\alpha^2\delta S_E^0}{S_N C S_N^0(\delta + \mu)} + (\frac{(1-\beta)\alpha}{S_N C} - \frac{\chi_2 S_N)(C - C^*)^2 b}{S_N C}\end{aligned}$$

For $\chi_1 = \frac{S_N C(C-C^*)^2 S_N^0}{(S_N - S_N^*)^2}$, and $\chi_2 = 1$, we have
$\frac{dL}{dt} < [-\mathcal{R}_0 b C^2 + \frac{(1-f)\alpha^2\delta S_E^0}{S_N^0(\delta+\mu)} + \frac{(1-\beta)\alpha S_N^2}{S_N C} - \frac{b}{C}](C - C^*)^2$,

Thus, $\frac{dL}{dt} < 0$ only if $\mathcal{R}_0 > 1$, and $\frac{dL}{dt} \leq 0$ if $C = C^*$, then by Theorem 2.2 the equilibrium EE is globally asymptotically stable.

□

5. Analysis of the sensitivity of the model parameters in relation to $\mathcal{R}_0$.

During the very rapid spread of the virus, a way of controlling it, is, trying to reduce it. Sensitivity analysis is one of the most effective ways to investigate the importance of each parameter separately in the transmission of the disease. Specifically, the definition of the normalized forward sensitivity index indicates the meaning of the variable associated with each parameter of the model to minimize the risk of any disease. Consequently, only the sensitivity analysis can determine how the ambiguity of the parameter is affected, we thus express a sensitivity coefficient as follows [17].

Definition 2. To examine the sensitivity of $\mathcal{R}_0$ to each of its parameters, the normalized forward sensitivity index concerning each parameter is computed below:

$$\Gamma_x^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial x}\frac{x}{\mathcal{R}_0}.$$

$$\begin{aligned}
\Gamma_f^{\mathcal{R}_0} &= -\frac{\alpha\delta S_E^0}{(1-f)\alpha\delta S_E^0 + (1-\beta)(\delta+\mu)S_N^0},\\
&= -\frac{f\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)}{(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)}.\\
\Gamma_\alpha^{\mathcal{R}_0} &= \frac{\alpha[2\alpha(1-f)\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)(\delta+\mu)\Gamma(\theta+\mu)]}{(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)}.\\
\Gamma_\delta^{\mathcal{R}_0} &= \frac{\delta(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)(\delta+\mu-1)}{(\delta+\mu)((1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu))}.\\
\Gamma_\beta^{\mathcal{R}_0} &= -\frac{\beta[\alpha(\delta+\mu)\Gamma(\theta+\mu)(\varphi+\beta+\mu+\theta)+(1-f)\alpha^2\delta\varphi\Gamma+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)]}{(\varphi+\beta+\mu+\theta)[(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)]}.\\
\Gamma_\Pi^{\mathcal{R}_0} &= 1.\\
\Gamma_\Gamma^{\mathcal{R}_0} &= \frac{\Gamma[-(1-f)\alpha^2(\beta+\mu+\theta)+(1-\beta)\alpha(\delta+\mu)(\theta+\mu)]}{(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)}.\\
\Gamma_\varphi^{\mathcal{R}_0} &= \frac{\varphi[(1-f)\alpha^2\delta\Gamma(\beta+\mu+\theta)-(1-\beta)\alpha(\delta+\mu)(\theta+\mu)\Gamma]}{(\varphi+\beta+\mu+\theta)[(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)]}.\\
\Gamma_\phi^{\mathcal{R}_0} &= \frac{\phi(\epsilon+\phi+\mu-1)}{(\epsilon+\phi+\mu)}.\\
\Gamma_\epsilon^{\mathcal{R}_0} &= \frac{\epsilon(\epsilon+\phi+\mu-1)}{(\epsilon+\phi+\mu)}.\\
\Gamma_\mu^{\mathcal{R}_0} &= \frac{\mu((1-f)\alpha^2\delta(1-\Gamma)+(1-\beta)\alpha\Gamma)(\epsilon+\phi+\mu))}{[(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)]} - \frac{\mu(r_1+2r_2\mu+3r_3\mu^2+4\mu^3)}{(\epsilon+\phi+\mu)(\delta+\mu)(\varphi+\beta+\mu+\theta)(\theta}\\
\Gamma_\theta^{\mathcal{R}_0} &= \frac{\theta[((1-f)\alpha^2\delta(1-\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma)(\varphi+\beta+\mu+\theta)(\theta+\mu)-((1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)(\varphi+\beta+2\mu+2\theta)]}{(\varphi+\beta+\mu+\theta)(\theta+\mu)[(1-f)\alpha^2\delta((1-\Gamma)(\varphi+\beta+\mu+\theta)+\varphi\Gamma)+(1-\beta)\alpha(\delta+\mu)\Gamma(\theta+\mu)]}.
\end{aligned}$$

Where

$$\begin{aligned}
r_1 &= (\epsilon+\phi)\delta(2\theta+\varphi+\beta)+(\delta+\epsilon+\phi)(\varphi+\beta+\theta)\theta.\\
r_2 &= (\epsilon+\phi)\delta+(\delta+\epsilon+\phi)(2\theta+\varphi+\beta)+(\varphi+\beta+\theta)\theta.\\
r_3 &= (\delta+\epsilon+\phi+2\theta+\varphi+\beta).
\end{aligned}$$

We have
$\Gamma_\alpha^{\mathcal{R}_0}, \Gamma_\delta^{\mathcal{R}_0}, \Gamma_\Pi^{\mathcal{R}_0}, \Gamma_\theta^{\mathcal{R}_0}, \Gamma_\phi^{\mathcal{R}_0}, \Gamma_\epsilon^{\mathcal{R}_0} > 0$ while, $\Gamma_f^{\mathcal{R}_0}, \Gamma_\beta^{\mathcal{R}_0}, \Gamma_\mu^{\mathcal{R}_0}, \Gamma_\varphi^{\mathcal{R}_0}, \Gamma_\Gamma^{\mathcal{R}_0} < 0$.

Table 2: Sensitivity index table

Symbol of the parameter	Index of sensitivity
β	-v
α	+v
μ	+v
Γ	-v
θ	+v
Π	+v
φ	-v
δ	+v
e_{NC}	-v
f	-v
ϵ	-v

This means that $\mathcal{R}_0$ increased in α, δ, Π, θ, ϕ and ϵ, when $\mathcal{R}_0$ decreases in f, β, μ, φ and Γ $\mathcal{R}_0$ does not depend on $\omega, r_s, \gamma, \eta, k_s, r_s, k_R, r_R$ and Λ, then $\Gamma_{\omega}^{\mathcal{R}_0} = 0, \Gamma_{r_s}^{\mathcal{R}_0} = 0, \Gamma_{\Lambda}^{\mathcal{R}_0} = 0, \Gamma_{k_s}^{\mathcal{R}_0} = 0, \Gamma_{r_R}^{\mathcal{R}_0} = 0, \Gamma_{k_R}^{\mathcal{R}_0} = 0$ and $\Gamma_{\eta}^{\mathcal{R}_0} = 0$.

6. Application of the model to optimal control

The optimal control for corruption consists of reducing the size of the exposed and corrupted components of the population to a minimum. The majority of authoritarians are willing to engage in corrupt tactics or enter into unjust agreements, which discourages honest citizens with few resources. Due to new technologies, it is not necessary to have a bunch of money or power to bring corruption to light and combat it, we can use social media and web-based platforms. In India, for example, a couple has created an anti-corruption website where people can report cases when they have been requested to pay bribes. An optimal control model for this was designed, using two controls below: ν_1: put corrupt individuals in jail and give a penalty. The media can inform and teach people about the damaging effects of corruption and, through public ex posure increase the political danger of those engaged in corrupt activities.
So we can word ν_2 as following:
ν_2: Combat corruption through the media and publicity for raising awareness.
By including both of the controls on the model (2), the optimal control model is given as follows:

$$\begin{cases} \dfrac{dS_N}{dt} &= \Gamma\Pi + \gamma(1-\theta)J - (\varphi + \beta + \mu + \theta + \nu_1)S_N - \alpha(1-\beta)(1-\nu_1)S_N C, \\ \dfrac{dS_E}{dt} &= (1-\Gamma)\Pi + (\varphi + \nu_1)S_N + r_s(1-k_s)R_s - (1-f)\alpha(1-\nu_2)S_E C - (\theta + \mu + \nu_1)S_E, \\ \dfrac{dE_C}{dt} &= (1-f)(1-\nu_2)\alpha S_E C - (\delta + \mu + \nu_1)E_C, \\ \dfrac{dC}{dt} &= (1-\beta)\alpha(1-\nu_2)S_N C + \alpha(\delta + \nu_1)E_C - (\epsilon + \phi + \mu + \nu_2)C, \\ \dfrac{dJ}{dt} &= (\phi + \nu_2)C - (\gamma\eta + \gamma(1-\theta) + \mu)J, \\ \dfrac{dR_s}{dt} &= \gamma\eta\Lambda J + (1-\sigma)(\epsilon + \nu_2)C - (r_s + \mu)R_s \end{cases} \tag{12}$$

There are two control factors ν_1 and ν_2 that minimize the optimal control model (1) given the objective function explained as such

$$J(\nu_1,\nu_2) = \int_{t_0}^{t_f} [A_1E_C + A_2E_{NC} + A_3C + \frac{1}{2}(\omega_1\nu_1^2 + \omega_2\nu_2^2)]dt, \tag{13}$$

where the terms $A_{i=1,2,3}$ are constants and are defined as the equilibrated cost factors of the corrupted and exposed members, respectively, and the weights of the values of each individual control measure are w_1 and w_2, the initial time is $t_0 = 0$ and the final time is t_f.
We evaluate the quadratic objective function because the intervention is non-linear, for more details see the connected works and citations [25, 42, 29]. Therefore, we aim to have an optimal control ν_1^*, ν_2^* such that

$$J\left(\nu_1^*,\nu_2^*\right) = \min\left\{J\left(\nu_1,\nu_2\right) \mid \left(\nu_1,\nu_2\right) \in U\right\},$$

where $U = \{(\nu_1,\nu_2) \mid \nu_i(t)$ is lebesgue measurable on $[0,t_f],\ 0 \le \nu_i(t) \le 1, i = 1,2\}$ is the closed set.

7. The hamiltonian and optimality system

Seem necessary for the optimal control to verify the condition imposed by the Pontryagin maximum principle. The above principle converts the equation system (12) and (13) into a problem of minimizing a pointwise Hamiltonian M, relative to ν_1 and ν_2. Both the Lagrangian $\mathcal{L}$ and the Hamiltonian M of the above optimal control system are defined as shown below

$$\mathcal{L} = A_1E_C + A_2E_{NC} + A_3C + \frac{1}{2}(\omega_1\nu_1^2 + \omega_2\nu_2^2),$$

and

$$\begin{aligned}
M = {} & A_1E_C + A_2E_{NC} + A_3C + \frac{1}{2}(\omega_1\nu_1^2 + \omega_2\nu_2^2) \\
& + \lambda_1[\Gamma\Pi + \gamma(1-\theta)J - (\varphi+\beta+\mu+\theta+\nu_1)S_N - \alpha(1-\beta)(1-\nu_1)S_NC] \\
& + \lambda_2[(1-\Gamma)\Pi + (\varphi+\nu_1)S_N + r_s(1-k_s)R_s - (1-f)\alpha(1-\nu_2)S_EC - (\theta+\mu+\nu_1)S_E] \\
& + \lambda_3[(1-f)(1-\nu_2)\alpha S_EC - (\delta+\mu+\nu_1)E_C] \\
& + \lambda_4[(1-\beta)\alpha(1-\nu_2)S_NC + \alpha(\delta+\nu_1)E_C - (\epsilon+\phi+\mu+\nu_2)C] \\
& + \lambda_5[(\phi+\nu_2)C - (\gamma\eta+\gamma(1-\theta)+\mu)J] \\
& + \lambda_6[\gamma\eta\Lambda J + (1-\sigma)(\epsilon+\nu_2)C - (r_s+\mu)R_s].
\end{aligned}$$

With $\lambda_i, i = 1,\dots,9$ be used as the adjoint variable functions that shall be found. Due to the convexity of the integrate of J. Regarding ν_1 and ν_2, the priory bounds in conditional solutions, and the Lipschitz criterion of the bound conditional system, then the existence of the optimal control has been proved through [26].

Theorem 7.1. *Assume that we have optimal controls u_1^*, $u*_2$ and $S_N, S_E, E_C, E_{NC}, C, J, R_s, R, H$ respective state system solutions that minimizes J on U, there are adjoint variables, $\lambda_1,\dots,\lambda_9$ from which*

$$\begin{aligned}
\tfrac{d\lambda_1}{dt} &= -\tfrac{\partial M}{\partial S_N} = \lambda_1((\varphi+\beta+\mu+\theta+\nu_1) + \alpha(1-\beta)(1-\nu_1)C) - \lambda_2(\varphi+\nu_1) - \lambda_4((1-\beta)\alpha(1-\nu_2)C, \\
\tfrac{d\lambda_2}{dt} &= -\tfrac{\partial M}{\partial S_E} = \lambda_2((1-f)\alpha(1-\nu_2)C + (\theta+\mu+\nu_1)) - \lambda_3(1-f)(1-\nu_2)\alpha C, \\
\tfrac{d\lambda_3}{dt} &= -\tfrac{\partial M}{\partial E_C} = \lambda_3(\delta+\mu+\nu_1) - \lambda_4\alpha(\delta+\nu_1), \\
\tfrac{d\lambda_4}{dt} &= -\tfrac{\partial M}{\partial C} = \lambda_1(\alpha(1-\beta)(1-\nu_1)S_N + \lambda_2(1-f)(1-\nu_2)S_E - \lambda_3((1-f)(1-\nu_2)\alpha S_E \\
&\qquad - \lambda_4((1-\beta)\alpha(1-\nu_2)S_N - (\epsilon+\phi+\mu+\nu_2)) - \lambda_5(\phi+\nu_2) - \lambda_6(1-\sigma)(\epsilon+\nu_2), \\
\tfrac{d\lambda_5}{dt} &= -\tfrac{\partial M}{\partial J} = -\lambda_1(\gamma(1-\theta) + \lambda_5(\gamma\eta+\gamma(1-\theta)+\mu) - \lambda_6\gamma\eta\Lambda, \\
\tfrac{d\lambda_6}{dt} &= -\tfrac{\partial M}{\partial R_s} = -\lambda_2 r_s(1-k_s) + \lambda_6(r_s+\mu).
\end{aligned}$$

With transversality conditions $\lambda_i(t_f) = 0$ *pour* $i = 1 \dots 9$.

Furthermore, the associated optimal controls ν_1^*, ν_2^* *are obtained from* $\frac{\partial M}{\partial \nu_1} = 0$, $\frac{\partial M}{\partial \nu_2} = 0$. *Thus, we got the characteristic equation in standard control arguments form involving the bounds on the controls as follows*
$\nu_1^* = \min\{1, \max(0, \Phi_1)\}$,
$\nu_2^* = \min\{1, \max(0, \Phi_2)\}$,
where
$\Phi_1 = \frac{\lambda_3 E_C - (\lambda_4 \alpha E_C + \lambda_1(\alpha(1-\beta)S_N C - S_N) + \lambda_2(S_N - S_E))}{\omega_1}$,
$\Phi_2 = \frac{\lambda_3(1-f)\alpha S_E C + \lambda_4((1-\beta)\alpha S_N - 1)C + \lambda_2(1-f)\alpha S_E C - (\lambda_5 + \lambda_6(1-\sigma))C}{\omega_2}$.

8. Numerical simulation and discussions

Analytical research will never be complete without numerical validation of the data. In the present section, we have presented some numerical simulations to follow the dynamics of the system (2) for various initial conditions and parameters given in Tables 3 and 4.

Thus to solve our model we have used the fourth-order Runge-Kutta method (RK4) in Matlab software. We took into account the parameters listed in Table 3 as well as the different values of the initial conditions given in Table 4.

By using this parameters we have calculated the reproduction number and we find that $\mathcal{R}_0 = 0.3865$. Wa have solved in this case our model by using this set of parameters and initial conditions and we present the results in graphical form. It is seen clearly from Figure 2 that the solution profiles of system (2) converges to the corruption-free equilibrium

$$\mathcal{E}_0 = (0.059 \times 10^7, 1.0535 \times 10^7, 0, 0, 0).$$

By replacing the value of α to 0.0002 we find $\mathcal{R}_0 = 3.8662 > 1$ then from Theorem 4.7 the Endemic Equilibrium is asymptotically stable, this result is shown in Figure 3. As indicated in Figure 4 we experimented the effect of changing the initial conditions of S_N and S_E, thus $\mathcal{R}_0 = 0.3865$ then we have the stability of S_N and S_E for the corruption-free equilibrium. Furthermore, it is seen from Figure 5 that the solution of (2) converges to the the endemic equilibrium

$$EE = (1.4978 \times 10^7, 0.9096 \times 10^7, 2.6159 \times 10^7, 2.6581 \times 10^7, 3.8366 \times 10^7, 2.5910 \times 10^7)$$

in all the three different initial values of $S_E(0)$ and $S_N(0)$. Figure 6 shows the stability of the solution of (2) in the three different values of $E_C(0)$ and $C(0)$, while the Figure 7 shows that the solution converges to the endemic equilibrium in the same three initial values of $E_C(0)$ and $C(0)$ with $\alpha = 0.0002$. It is clear from Figure 8 that the solution of (2) is stable and converges to the corruption-free equilibrium in all the three different initial values of individuals $J(0)$ and $R_s(0)$, while in Figure 9, with the same initial values of $J(0)$ and $R_s(0)$ and $\alpha = 0.0002$, so that $\mathcal{R}_0 = 3.8662$ then the solution converges to the endemic equilibrium.

Finally, all results of this section support the theoretical results of the local and the global asymptotic stability of corruption-free and endemic equilibrium presented in the previous sections.

9. Conclusion

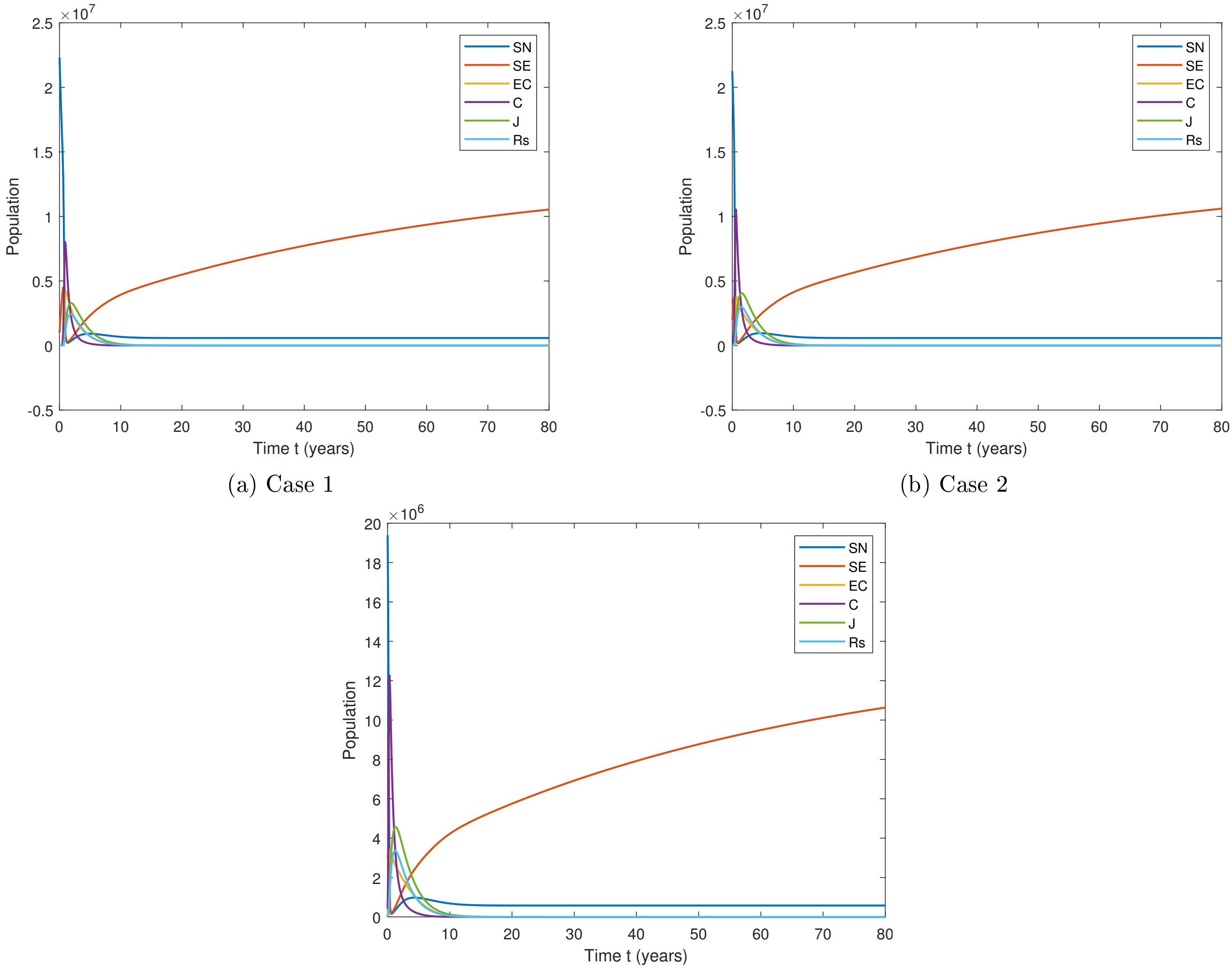
In this paper, we suggest a novel mathematical model that considers corruption as an infectious disease. This model is more general than the ones which exist in the literature today. The dynamics of the interaction between the compartments are described mathematically by a system of nine ODEs. This paper attempts to prove the existence and uniqueness of the solution to our problem via the fixed point theorem. The generation matrix is used to determine the basic reproduction number $\mathcal{R}_0$. The question of the stability of equilibrium points of the model is well examined throughout the paper. On the one hand, there is that which is asymptotically stable and that which is globally stable. Also, the paper highlights the importance of each parameter for the transmission of corruption by applying the concept of normalized forward sensitivity.

Table 3: Basic values for Parameters of system (2)

Parameter	Value	Source
μ	$0.011 \leq \mu \leq 0.021$	[12]
p	0.036	Assumed
θ	0.000001	[12]
σ	(0,1)	Varied
φ	0.02	Assumed
f	(0.1)	Varied
ω	0.001	Assumed
τ	(0.1)	Varied
η	0.0001	Assumed
Π	30.000	[12]
Λ	0.01	Assumed
e_{NC}	0.02	Assumed
ψ	0.143	[12]
r_R	0.02	Assumed
e	(0.1)	Varied
r_s	0.01	Assumed
k_s	0.001	Assumed
ϕ	0.0001 τ	[12]
k_R	0.01	Assumed

Table 4: Initial values of variables of system (2)

Initial values	Case 1	Case 2	Case 3
N	2330769	2330769	2330769
$S_N(0)$	2229903	2082863	1104763
$S_E(0)$	100000	200000	300000
$E_C(0)$	100	500	1000
$C(0)$	10	3000	400000
$J(0)$	50	400	2000
$R_s(0)$	100	1000	7000

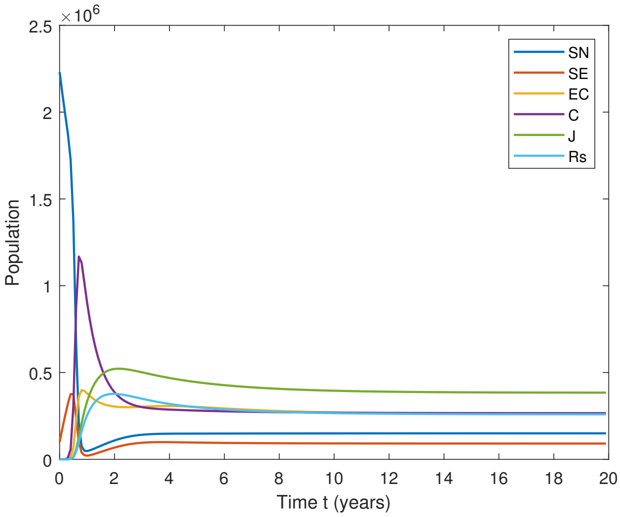


(a) Case 1

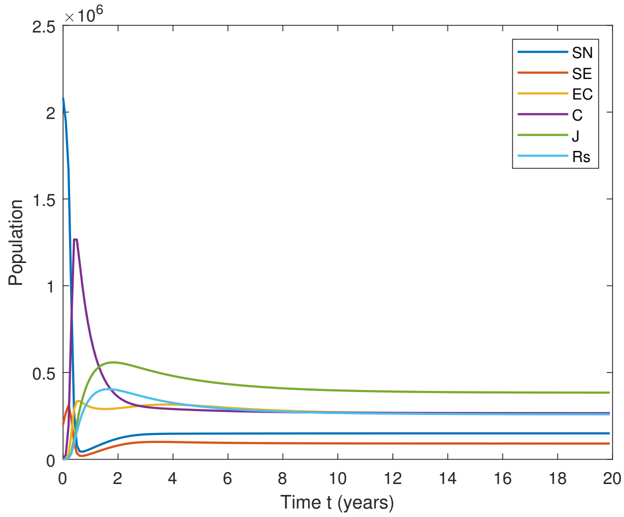
(b) Case 2

(c) Case 3

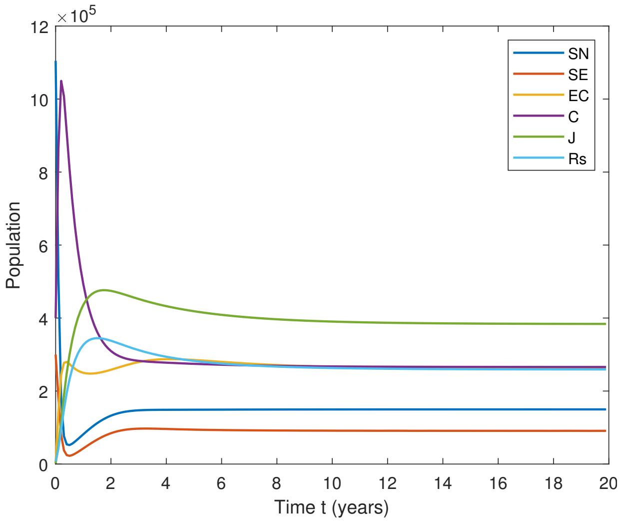
Figure 2: The graphical representation of the model solution in the first case. This figure shows that the corruption-free equilibrium of system (2) is $(0.059 \times 10^7, 1.0535 \times 10^7, 0, 0, 0)$.



(a) Case 4

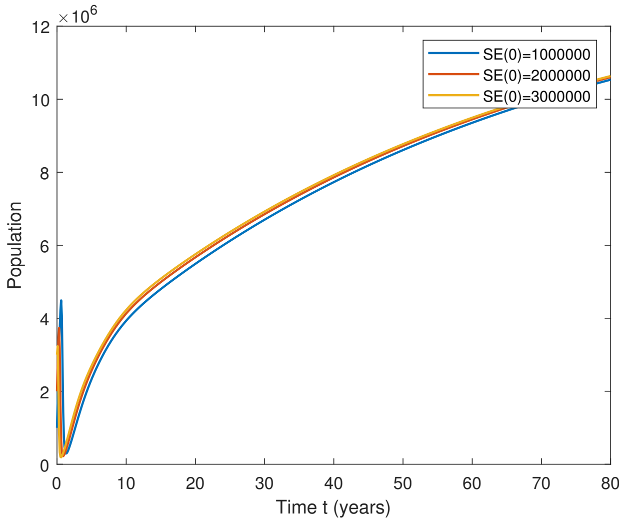


(b) Case 5

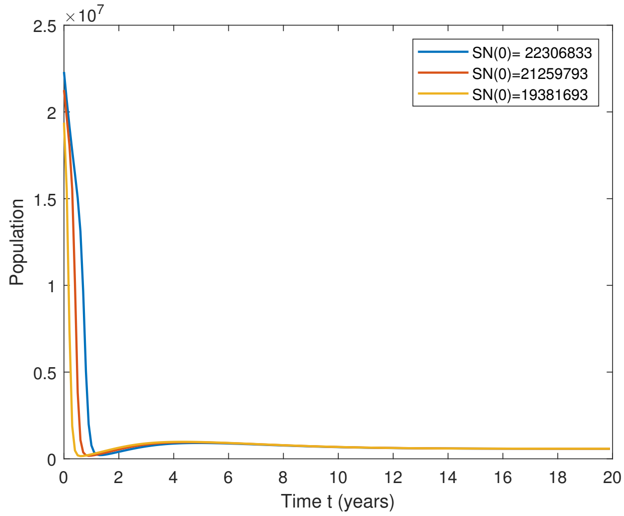


(c) Case 6

Figure 3: The graphical representation of the model solution in the second case. This figure shows that the endemic equilibrium of system (2) is $(1.4978 \times 10^7, 0.9096 \times 10^7, 2.6159 \times 10^7, 2.6581 \times 10^7, 3.8366 \times 10^7, 2.5910 \times 10^7)$.

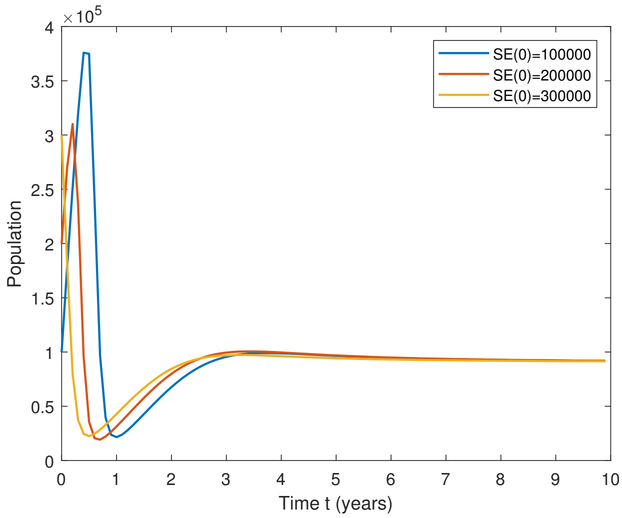


(a) Case 7

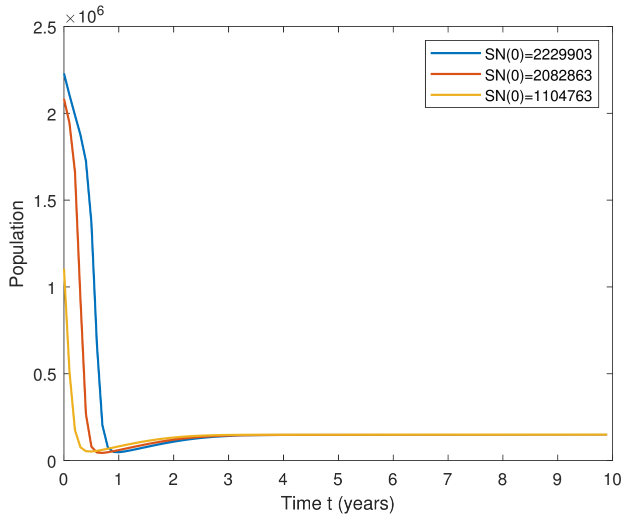


(b) Case 8

Figure 4: Numericals solutions of the model for parameters and different initial conditions of S_N and S_E given in Tables 3 and 4, here $\mathcal{R}_0 = 0.3865$ and the stability is for the corruption-free equilibrium .

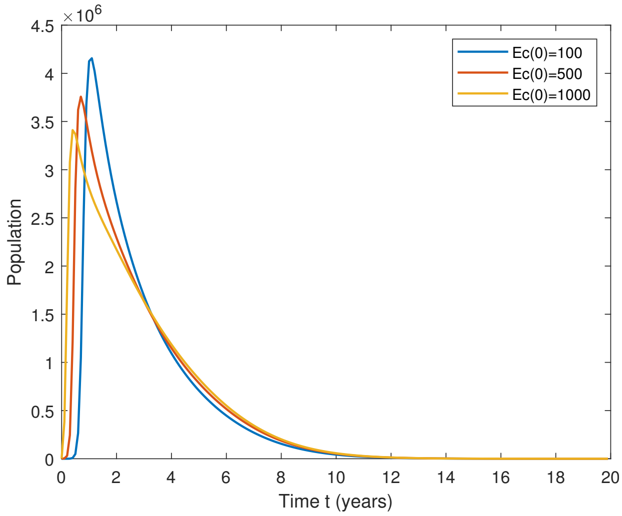


(a) Case 9

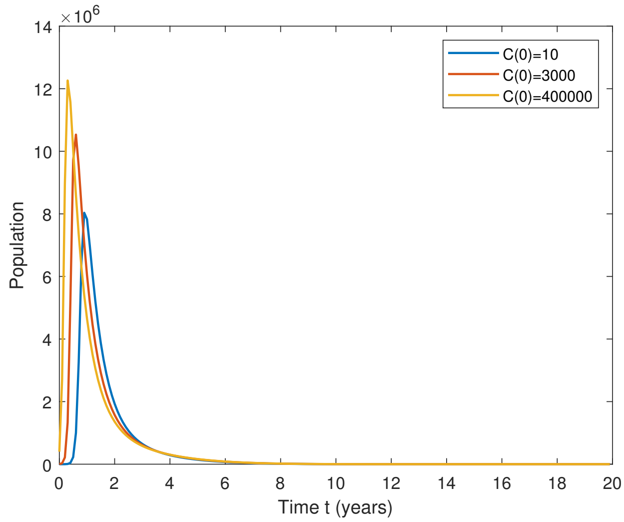


(b) Case 10

Figure 5: Numericals solutions of the model (2) for parameters and different initial conditions of S_N and S_E given in Tables 3 and 4, here $\mathcal{R}_0 = 3.8662$ and the stability is for the endemic equilibrium .

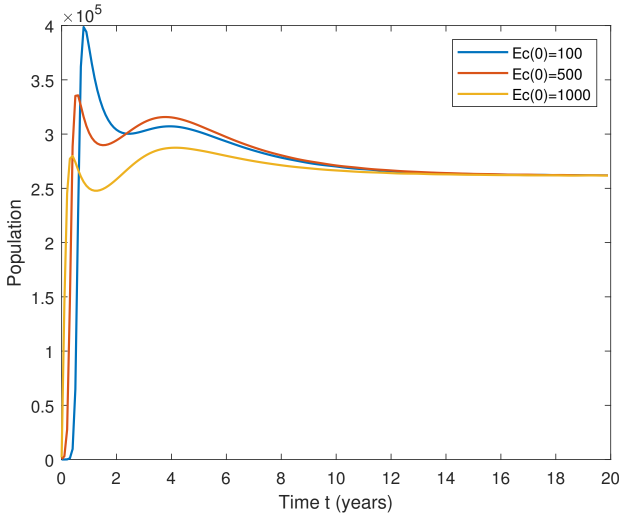


(a) Case 11

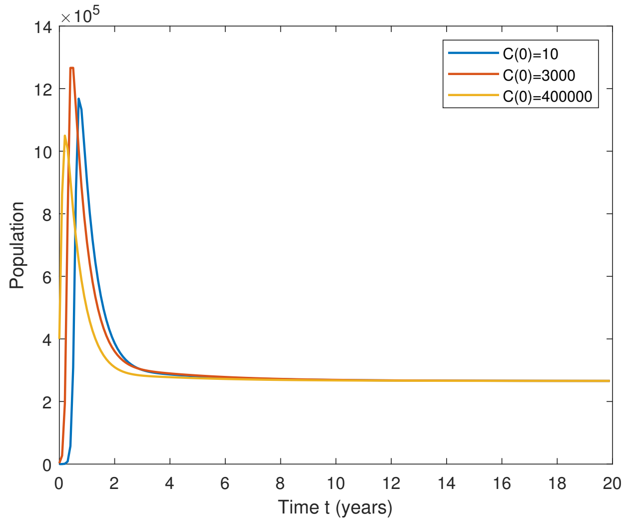


(b) Case 12

Figure 6: Numericals solutions of the model (2) for parameters and different initial conditions of E_C and C given in Tables 3 and 4, here $\mathcal{R}_0 = 0.3865$ and the stability is for the corruption-free equilibrium .

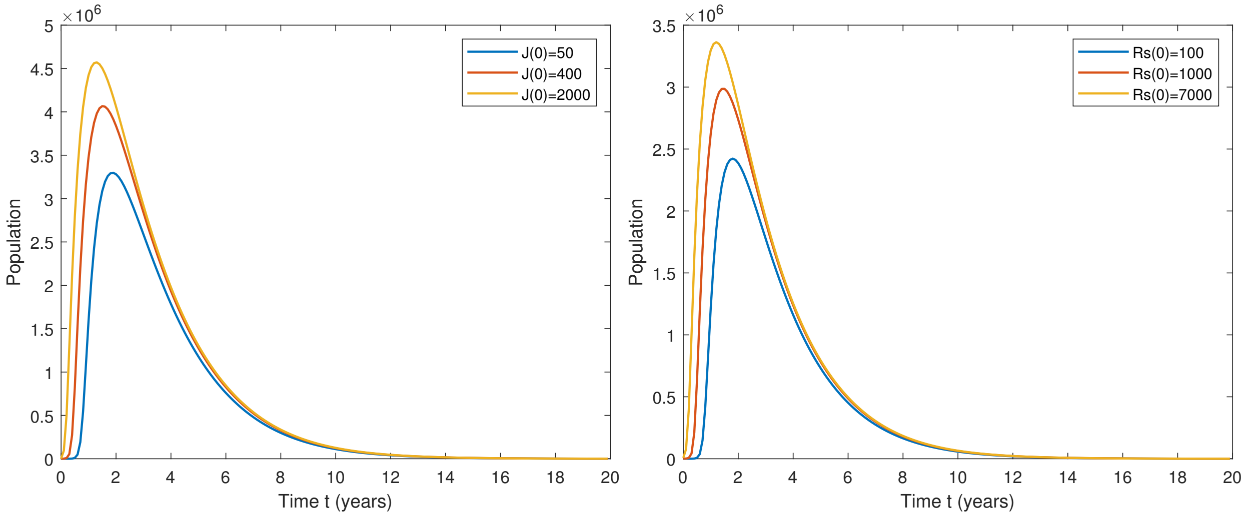


(a) Case 13



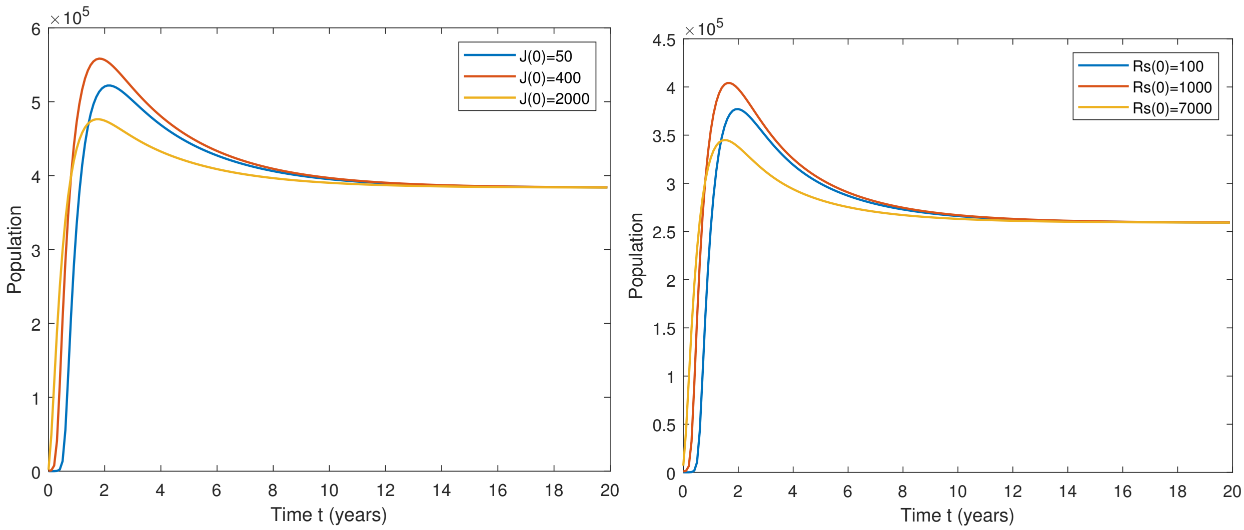
(b) Case 14

Figure 7: Numericals solutions of the model (2) for parameters and different initial conditions of E_C and C given in Tables 3 and 4, here $\mathcal{R}_0 = 3.8662$ and the stability is for the corruption-free equilibrium.



(a) Case 14 (b) 15

Figure 8: Numericals solutions of the model (2) for parameters and different initial conditions of J and R_s given in Tables 3 and 4, here $\mathcal{R}_0 = 0.3865$ and the stability is for the corruption-free equilibrium .



(a) Case 16 (b) 17

Figure 9: Numericals solutions of the model (2) for parameters and different initial conditions of J and R_s given in Tables 3 and 4, here $\mathcal{R}_0 = 3.8662$ and the stability is for the endemic equilibrium .

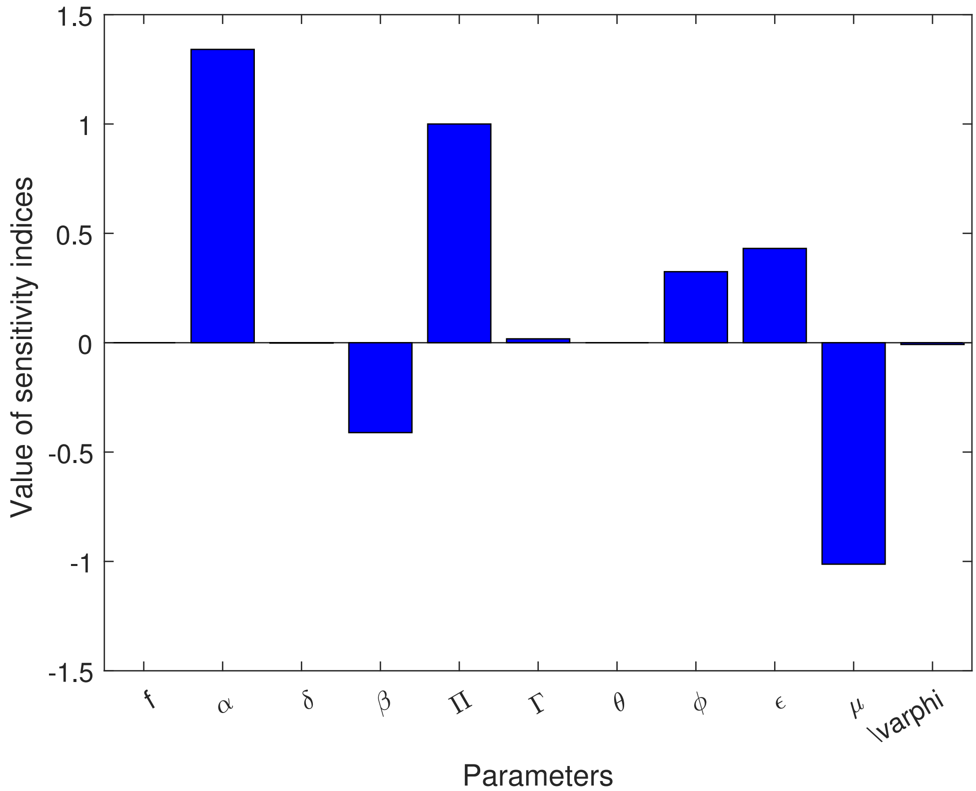


Figure 10: Sensitivity analysis of $\mathcal{R}_0$ with respect to model parameters

Likewise, the theory of optimal control is employed to investigate under what conditions the diffusion of corruption can be successfully controlled and to study the effect of a possible combination of both controls on the corruption's transmission. Finally, the analytical result is verified by using digital simulations.

References

[1] N. Özdemir ve E. Uçar, "Investigating of an immune system-cancer mathematical model with Mittag-Leffler kernel", AIMS Mathematics, vol. 5, no. 2, pp. 1519–1531, (2020).

[2] U. Sümeyra (2021), Existence and uniqueness results for a smoking model with determination and education in the frame of non-singular derivatives, Discrete and Continuous Dynamical Systems Series S, 14(7).

[3] U. Esmehan, U. Sümeyra, E. Firat, Ö. Neçati, A Fractional SAIDR Model in the Frame of Atangana-Baleanu Derivative, Fractal and Fractional, 5(2), (2021).

[4] U. Esmehan, U. Sümeyra, E. Firat, Ö. Neçati, Investigation of E-Cigarette Smoking Model with Mittag-Leffler Kernel, Foundations of Computing and Decision Sciences, 46(1), 97-109, (2021).

[5] E. Firat, U. Sümeyra, Ö. Neçati, System Analysis of HIV Infection Model with CD4T under Non-Singular Kernel Derivative, Applied Mathematics and Nonlinear Sciences, 5(1), 139-146, (2020).

[6] S. Uçar, Existence Results for a Computer Virus Spreading Model with Atangana-Baleanu Derivative, Celal Bayar University Journal of Science , 17 (1) , 67-72, (2020).

[7] E. Uçar, N. Özdemir, A fractional model of cancer-immune system with Caputo and Caputo–Fabrizio derivatives, The European Physical Journal Plus, vol. 136, (2021).

[8] U. Sümeyra, Analysis of a basic SEIRA model with Atangana-Baleanu derivative, AIMS Mathematics, 5(2), 1411-1424, (2020).

[9] A. Alla Hamou, E. Azroul, & A. Lamrani Alaoui, Fractional model and numerical algorithms for predicting covid-19 with isolation and quarantine strategies. International Journal of Applied and Computational Mathematics, 7(4), 1-30, (2021).

[10] A.A. Hamou, E. Azroul, Z. Hammouch, & A.L. Alaoui, A fractional multi-order model to predict the COVID-19 outbreak in Morocco. Appl. Comput. Math, 20(1), 177-203, (2020).

[11] A. Allahamou, E. Azroul, Z. Hammouch, & A.L. Alaoui, Modeling and numerical investigation of a conformable co-infection model for describing Hantavirus of the European moles. Mathematical Methods in the Applied Sciences, 45(5), 2736-2759, (2022).

[12] S. Abdulrahman, Stability analysis of the transmission dynamics and control of corruption, Pacific Journal of Science and Technology, vol. 15, no. 1, pp. 99–113, (2014).

[13] Alemneh, Haileyesus Tessema. Mathematical Modeling, Analysis, and Optimal Control of Corruption Dynamics . Journal of Applied Mathematics 2020 (1 août 2020): 1-13. https://doi.org/10.1155/2020/5109841.

[14] V. Akinsola, et A. Binuyo, Stability Analysis of the Corruption Free Equilibrium of the Mathematical Model of Corruption in Nigeria, Mathematical Journal of Interdisciplinary Sciences 8 (30 mars 2020): 61-68. https://doi.org/10.15415/mjis.2020.82008.

[15] N. Bacaër, A Short History of Mathematical Population Dynamics, London: Springer London, 2011. https://doi.org/10.1007/978-0-85729-115-8.

[16] *Baum, Anja, Clay Hackney, Paulo Medas, et Mouhamadou Sy. Governance and State-Owned Enterprises: How Costly Is Corruption? IMF Working Papers 2019, no 253 (22 novembre 2019). https://doi.org/10.5089/9781513519296.001.*

[17] N. Chitnis, J.M. Hyman, J.M. Cushing, Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bull Math Biol. 2008 Jul;70(5):1272-96. doi: 10.1007/s11538-008-9299-0. Epub 2008 Feb 22. PMID: 18293044.

[18] A. Cuervo-Cazurra, Corruption in International Business . Journal of World Business 51, no 1 (janvier 2016): 35-49. https://doi.org/10.1016/j.jwb.2015.08.015.

[19] K. Dietz, The Estimation of the Basic Reproduction Number for Infectious Diseases . Statistical Methods in Medical Research 2, no 1 (mars 1993): 23-41. https://doi.org/10.1177/096228029300200103.

[20] S. Athithan, M. Ghosh, X.Z. Li, Mathematical modeling and optimal control of corruption dynamics. Asian-Eur J Math 11(6) (2018).

[21] K. Dietz, D. Schenzle, Proportionate mixing models for age-dependent infection transmission, J. Math. Biology 22, 117–120 (1985). https://doi.org/10.1007/BF00276550.

[22] E.X. DeJesus, C. Kaufman, Routh-Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations. Physical Review A, 35(12), 5288, (1987),

[23] Louis I.Dublin, Alfred J. Lotka, On the True Rate of Natural Increase, Journal of the American Statistical Association 20, no 151 (septembre 1925): 305. https://doi.org/10.2307/2965517.

[24] Education Manual, The conceptual basis of the fight against corruption, 2014.

[25] F.B. Agusto, M.A. Khan, Optimal control strategies for dengue transmission in pakistan, Mathematical Biosciences, Vol 305, 2018, 102-121, ISSN 0025-5564, https://doi.org/10.1016/j.mbs.2018.09.007.

[26] W.H. Fleming, R.W. Rishel, Deterministic and stochastic optimal control (Vol. 1). Springer Science & Business Media, (2012).

[27] M.H. Gail, P.S. Rosenberg. Perspectives on Using Back calculation to Estimate HIV Prevalence and project AIDS incidence . in AIDS Epidemiology, 1-38. Boston, 1992. https://doi.org/10.1007/978-1-4757-1229-2-1.

[28] J.A.P. Heesterbeek, K. Dietz, The Concept of R_0 in Epidemic Theory, Statistica Neerlandica 50, no 1 (mars 1996): 89-110. https://doi.org/10.1111/j.1467-9574.1996.tb01482.x.

[29] H. Sofia and F. Rodrigues , "Optimal Control and Numerical Optimization Applied to Epidemiological Models", Applications of Mathematics, 2012.

[30] I. Yusuf, S. Abdulrahman, B. Musa and G. Adamu, "Controlling the Spread of Corruption through Social Media: A Mathematical Modelling Approach", Nigeria 2016.

[31] K.W. Ogilvy and A.G. McKendrick, A contribution to the mathematical theory of epidemicsProc. R. Soc. Lond. A115700–721 http://doi.org/10.1098/rspa.1927.0118.

[32] R.R. Ktjczynski, The balance of births and deaths, s. d., 8. 1928.

[33] L. Lemecha, S. Feyissa, Mathematical modeling and analysis of corruption dynamics. Ethiopian Journal of Science and Sustainable Development, 5(2), 13-25, (2018).

[34] G. MACDONALD, The analysis of equilibrium in malaria. Trop Dis Bull. 1952 Sep;49(9):813-29. PMID: 12995455.

[35] N.O. Mokaya, H.T. Alemmeh, C.G.Ngari, G.G. Muthuri, Mathematical Modelling and Analysis of Corruption of Morals amongst Adolescents with Control Measures in Kenya . Édité par Chun Wei. Discrete Dynamics in Nature and Society 2021 (21 avril 2021): 1-16. https://doi.org/10.1155/2021/6662185.

[36] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Mathematical Biosciences, Vol 180, Issues 1–2, 2002, 29-48, ISSN 0025-5564, https://doi.org/10.1016/S0025-5564(02)00108-6.

[37] S. Oluwaseun, T. Malik, Optimal Control in Epidemiology . Annals of Operations Research 251, no 1-2 (avril 2017): 55-71. https://doi.org/10.1007/s10479-015-1834-4.

[38] D. Smith, N. Keyfitz, Mathematical demography: selected papers. Biomathematics, vol. 6. Berlin: Springer, 1977.

[39] F.R. Sharpe, A Problem in Age-Distribution , s. d., 2., 435–438.

[40] K.N. Trirogoff, L.S. Pontryagin Selected Works: The Mathematical Theory of Optimal Processes (L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, & L.W. Neustadt, Eds.) (1st ed.), Routledge, (2000) https://doi.org/10.1201/9780203749319.

[41] T. Vito, International monetary fund "corruption around the world: Causes, consequences, scope and cures" Fiscal Affairs Department, 1998.

[42] S. Ullah, Muhammad Altaf Khan, & J.F. Gómez-Aguilar. "Mathematical Formulation of Hepatitis B Virus with Optimal Control Analysis" . Optimal Control Applications and Methods 40, no 3 (mai 2019): 529-44.

https://doi.org/10.1002/oca.2493.

[43] J.A. Yorke, & al. "Dynamics and Control of the Transmission of Gonorrhea," Sexually Transmitted Diseases, vol. 5, no. 2, 1978, pp. 51–56. JSTOR, http://www.jstor.org/stable/44966942. Accessed 20 Jul. 2022.

[44] C. Chicone, Ordinary Differential Equations with Applications, Texts in Applied Mathematics. Springer New York, (2006). https://doi.org/10.1007/0-387-35794-7