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# Existence Results for Nonautonomous Impulsive Fractional Evolution Equations

Dimplekumar Chalishajar[a], Duraisamy Senthil Raja[b], Kulandhaivel Karthikeyan[b], Ponnusamy Sundararajan[c]

[a] *Department of Applied Mathematics, 435 Mallory Hall, Virginia Military Institute, Lexington, VA-24450, USA.*
[b]*Department of Mathematics, K. S. Rangasamy College of Technology, Tiruchengode-637 215, Tamil Nadu, India.*
[c]*Department of Mathematics, Arignar Anna Government Arts College, Namakkal-637 002, Tamil Nadu, India.*

**Abstract**

In this paper, we investigate the mild solutions of a nonlocal Cauchy problem for nonautonomous fractional evolution equations New results are obtained by using Sadovskii's fixed point theorem and the Banach contraction mapping principle. An example is given to illustrate the theory.

*Keywords:* Nonlocal condition, Fractional nonautonomous differential equations, Measure of noncompactness, Mild solution, Impulsive Fractional evolution integrodifferential equations.
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## 1. Introduction

The qualitative behavior of evolutionary fractional differential and difference equations, whose right-hand side is explicitly time-dependent can be described by nonautonomous dynamics. Over recent years, the theory of such systems has developed into a highly active field related to, yet recognizably distinct from that of classical autonomous dynamical systems. This development was motivated by problems of applied mathematics, in particular in the life sciences where genuinely nonautonomous systems abound. On the other hand, the existence of the solution of the fractional differential equations with nonlocal conditions has been investigated widely by many authors as, the nonlocal conditions are more realistic than the classical initial conditions such as in dealing with many physical problems.

In recent years, impulsive differential equations have become an active area of research due to their demonstrated applications in wide spread fields of science and engineering such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and many others. Many physical

*Email addresses:* dipu17370@gmail.com (Dimplekumar Chalishajar), dsr.senthil@gmail.com (Duraisamy Senthil Raja), karthi_phd2010@yahoo.co.in (Kulandhaivel Karthikeyan), ponsundar03@gmail.com (Ponnusamy Sundararajan)

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systems which are characterized by the occurrence of an abrupt change in the state of the system can be described by impulsive differential equations. These changes occur at certain time instants over a period of negligible duration. Impulsive differential equations are also an appropriate model to hereditary phenomena for which a delay argument arises in the modeling equations. These equations describe the evolution processes that are subjected to abrupt changes and discontinuous jumps in their states. Many physical systems like the function of pendulum clock, the impact of mechanical systems, preservation of species by means of periodic stocking or harvesting and the heartâĂŹs function, etc. naturally experience the impulsive phenomena. Similarly in many other situations, the evolution processes have the impulsive behavior. For example, the interruptions in cellular neural networks, the damperâĂŹs operation with percussive effects, electromechanical systems subject to relaxation oscillations, dynamical systems having automatic regulations, etc., have the impulsive phenomena. The existence, uniqueness and stability of mild solutions to functional differential equations with impulsive conditions have been considered by many authors in literature (refer [2,3,29,30]).

During the past decades, the fractional differential equations have been proven to be valuable tools in the investigation of many phenomena in engineering and physics, they attracted many researchers (cf., e.g., [1,4,9,10,17,18,23,25,28]). Fractional derivatives introduce amazing instrument for the description of general properties of different materials and processes. This is the primary advantage of fractional derivatives in comparison with classical integer order models, in which such impacts are in fact ignored. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids, rocks and in many other fields (see [1, 4]). Since fractional order differential equations play important roles in modeling real world problems related to biology, viscoelasticity, physics, chemistry, control theory, economics, signal and image processing phenomenon, bioengineering, and so forth (for details, see [5-7,9-10]), it is investigated that fractional order differential equations model real world problems more accurately than differential equations of integer order. On the other hand, the autonomous and nonautonomous evolution equations and related topics were studied in, e.g., [8,13-16,15,18,21,23,26,31-35], and the nonlocal Cauchy problem was considered in, e.g., [4,7,11,17,19,20,22,24,33]. Besides from the aforesaid problems, recently by using fixed point theory, several remarkable problems have been investigated in FDEs with various boundary conditions, for detail see [14] and the references therein.

In this paper, we consider the following nonlocal Cauchy problem for nonautonomous fractional evolution equations:

$$\begin{cases} \frac{d^q u(t)}{dt^q} & = & -A(t)u(t) + f(t, (K_1u)(t), (K_2u)(t), \dots, (K_nu)(t), t \in I = [0, T] \\ \Delta y|_{t=t_k} & = & I_k(y(t_k^-)), t = t_k, k = 1, 2, \dots, m, \\ u(0) & = & A^{-1}(0)g(u) + u_0; \end{cases} \tag{1.1}$$

where $0 < q < 1,\ T > 0, g : C(I; X) \to X$. The terms $(K_iu)(t), i = 1, 2, \dots, n$ are defined by

$$(K_iu)(t) = \int_0^t k_i(t, s)u(s)ds,$$

the positive functions $k_i(t, s)$ are continuous on $D = \{(t, s) \in R^2 : 0 \le s \le t \le T\}$ and

$$K_i^* = \sup_{t\in[0,t]} \int_0^t k_i(t, s)ds < \infty.$$

Throughout this work, we set $I = [0, T]$. We denote by $X$ a Banach space, $L([0, T]; X) := L(X)$ the space of all linear and bounded operators on $X$, and $C(I, X)$ the space of all $X$-valued continuous functions on $I$.

Let us assume that $u \in L(X)$ and $A(t)$ is a family of bounded linear operators defined in a Banach space $X$. The fractional order integral of the function $u$ is understood here in the Riemann-Liouville sense, i.e.,

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}u(s)ds.$$

In this paper, we denote that $C$ is a positive constant, and assume that a family of closed linear operators $\{A(t) : t \in [0,t]\}$ satisfying the followings:

(A1) The domain $D(a)$ of $\{A(t) : t \in [0,T]\}$ is dense in the Banach space $X$, $A(t)$ depends on $t$.

(A2) The operator $[A(t)+\lambda]^{-1}$ exists in $L(X)$ for any $\lambda$ with $Re\lambda \leq 0$ and

$$\|[A(t)+\lambda]^{-1}\| \leq \frac{C}{|\lambda+1|}, t \in [0,T], \ C \text{ is a real constant}$$

(A3) There exist constants $\gamma \in (0,1]$ and $C$ such that

$$\|[A(t_1)-A(t_2)]A^{-1}(s)\| \leq C|t_1-t_2|^{\gamma}; \ t_1, t_2, s \in [0,T].$$

Under condition (A2), each operator $-A(s), s \in [0,T]$, generates an analytic semi-group $\exp(-tA(s)), t > 0$, and there exists a constant $C$ such that

$$\|A^n(s)\exp(-tA(s))\| \leq \frac{C}{t^n},$$

where $n = 0, 1, t > 0, s \in [0,T]$, (refer [13]).

We study the existence of mild solution of (1.1) and obtain the existence theorem based on the measures of noncompactness. An example is given to show an application of the abstract results.

## 2. Preliminaries

**Definition 2.1.** The fractional order integral of the function $h \in L^1([a,b],R)$ of order $\alpha \in R_+$ is defined by

$$I_a^{\alpha}h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds$$

where $\Gamma$ is the Gamma function.

**Definition 2.2.** For a function $h$ given on the interval $[a,b]$, the $\alpha$th Riemann-Liouville fractional order derivative of $h$, is defined by

$$(D_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds,$$

here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of $\alpha$.

**Definition 2.3.** For a function $h$ given on the interval $[a,b]$, the Caputo fractional order derivative of $h$, is defined by

$$({}^cD_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of $\alpha$.

**Lemma 2.4.** *Let $\alpha > 0$, then the differential equation ${}^cD^{\alpha}h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

*where $c_i \in R$, $i = 0, 1, 2, \cdots, n-1, n = [\alpha]+1$.*

**Lemma 2.5.** *Let $\alpha > 0$, then*

$$I^{\alpha}({}^cD^{\alpha}h)(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

*for some $c_i \in R, i = 0, 1, 2, \cdots, n-1$, $n = [\alpha]+1$.*

**Lemma 2.6.** *([28])*
*(1)* $I^q : L^1[0,T] \to L^1[0,T]$;
*(2) For* $g \in L^1[0,T]$, *we have*

$$\int_0^t \int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\gamma-1} g(s)dsd\eta = \mathcal{B}(q,\gamma) \int_0^t (t-s)^{q+\gamma-1} g(s)ds,$$

*where* $\mathcal{B}(q,\gamma)$ *is a Beta function.*

**Definition 2.7.** Let $B$ be a bounded set of semi-normed linear space $Y$. The Kuratowski's measure of noncompactness (for brevity, $\alpha$-measure) of $B$ is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \leq d\}.$$

From the definition we can get some properties of $\alpha$-measure immediately, see ([5]).

**Lemma 2.8.** *([5]) Let* $A$ *and* $B$ *be bounded sets of* $X$. *Then*
*(1)* $\alpha(A) \leq \alpha(B)$, *if* $A \subseteq B$.
*(2)* $\alpha(A) = \alpha(A^{cl})$, *where* $A^{cl}$ *denotes the closure of* $A$.
*(3)* $\alpha(A) = 0$ *if and only if* $A$ *is precompact.*
*(4)* $\alpha(\lambda A) = |\lambda|\alpha(A),\ \lambda \in R$.
*(5)* $\alpha(A \cup B) = max\{\alpha(A), \alpha(B)\}$.
*(6)* $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, *where* $A + B = \{x + y : x \in A,\ y \in B\}$.
*(7)* $\alpha(A + x_0) = \alpha(A)$, *for any* $x_0 \in X$.
*For* $H \subset C(I,X)$, *we define*

$$\int_0^t H(s)ds = \Big\{ \int_0^t u(s)ds : u \in H \Big\},$$

*where* $H(s) = \{u(s) \in X : u \in H\}$, *for* $t \in I$.

The following lemma will be needed:

**Lemma 2.9.** *([5]) If* $H \subset C(I,X)$ *is a bounded, equicontinuous set, then*
*(1)* $\alpha(H) = sup_{t\in I}\alpha(H(t))$.
*(2)* $\alpha(\int_0^t H(s)ds) \leq \int_0^t \alpha(H(s))ds$, *for* $t \in I$.

**Lemma 2.10.** *([12]) If* $\{u_n\}_{n=1}^{\infty} \subset L^1(I,X)$ *and there exists a* $m(.) \in L^1(I,R^+)$ *such that*

$$\|u_n(t)\| \leq m(t), \quad a.e \quad t \in I,$$

*then* $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ *is integrable and*

$$\alpha\Big(\Big\{ \int_0^t u_n(s)ds \Big\}_{n=1}^{\infty}\Big) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds.$$

We use the following Sadovskii's fixed point theorem:

**Definition 2.11.** ([27]) Let $P$ be a operator in Banach space $X$. If $P$ is continuous and takes bounded sets into bounded sets, and $\alpha(P(H)) < \alpha(H)$ for every bounded set $H$ of $X$ with $\alpha(H) > 0$, then $P$ is said to be a condensing operator on $X$.

**Lemma 2.12.** *(Sadovskii's fixed point theorem [27]) Let* $P$ *be a condensing operator on Banach space* $X$. *If* $P(B) \subseteq B$ *for a convex, closed and bounded set* $B$ *of* $X$, *then* $P$ *has a fixed point in* $B$.

According to [10], a mild solution of (1.1) can be defined as follows:

**Definition 2.13.** A function $u \in C(I, X)$ satisfying the equation

$$\begin{aligned} u(t) &= A^{-1}(0)g(u) + u_0 + \int_0^t \psi(t-\eta,\eta)U(\eta)A(0)[A^{-1}(0)g(u)+u_0]d\eta \\ &+ \int_0^t \psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))d\eta \\ &+ \int_0^t\int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))dsd\eta \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)) \end{aligned}$$

is called a mild solution of (1.1), where

$$\psi(t,s) = q\int_0^\infty \theta t^{q-1}\xi_q(\theta)\exp(-t^q\theta A(s))d\theta$$

and $\xi_q$ is a probability density function defined on $[0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\sigma x}\xi_q(\sigma)d\sigma = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1+qj)}, q\in(0,1], x>0,\ \Gamma \text{ is a gamma function}$$

and

$$\varphi(t,\tau) = \sum_{k=1}^\infty \varphi_k(t,\tau),$$

where

$$\begin{aligned} \varphi_1(t,\tau) &= [A(t)-A(\tau)]\psi(t-\tau,\tau), \\ \varphi_{k+1}(t,\tau) &= \int_\tau^t \varphi_k(t,s)\varphi(s,\tau)ds, k=1,2,\ldots, \end{aligned}$$

and

$$U(t) = -A(t)A^{-1}(0) - \int_0^t \varphi(t,s)A(s)A^{-1}(0)ds.$$

To our purpose the following conclusions will be needed:

**Lemma 2.14.** *([10]) The operator-valued functions $\psi(t-\eta,\eta)$ and $A(t)\psi(t-\eta,\eta)$ are continuous in uniform topology in the variables $t,\eta$, where $0\le\eta\le t-\varepsilon, 0\le t\le T$, for any $\varepsilon>0$. Clearly,*

$$\|\psi(t-\eta,\eta)\| \le C(t-\eta)^{q-1}. \tag{2.1}$$

*Moreover, we have*

$$\|\varphi(t,\eta)\| \le C(t-\eta)^{\gamma-1}. \tag{2.2}$$

**Remark 2.15.** From the proof of Theorem 2.5 in [10], we can see (1) $\|U(t)\| \le C + Ct^\gamma$.
(2) For $t\in I$, $\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta$ is uniformly continuous in the norm of $L(X)$ and

$$\Big\|\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta\Big\| \le C^2t^q\Big(\frac{1}{q}+t^\gamma B(q,\gamma+1)\Big) := \widetilde{M(t)}. \tag{2.3}$$

## 3. Existence of solution

Assume that

**(B1)** $f$ satisfies caratheödory property, i.e., $f : I\times X\times X\times\cdots\times X \to X$ satisfies $f(.,v_1,v_2,...,v_n) : I \to X$ is measurable for all $v_i \in X,\ i = 1,2,\ldots,n$ and $f(t,.,.,\ldots,.) : X \times X \times \cdots \times X \to X$ is continuous for $a.e\ t \in I$.

Also, there exists a positive function $\mu(.) \in L^p(I,R^+)(p > \frac{1}{q} > 1)$ and a continuous nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ such that

$$\|f(t,v_1,v_2,...,v_n)\| \le \mu(t)\omega\Big(\sum_{i=1}^{n}\|vi\|\Big),\quad (t,v_1,v_2,...,v_n) \in I \times X \times X \times \cdots \times X,$$

and set $T_{p,q}$ =max$\{T^{q-\frac{1}{p}},T^q\}$.

**(B2)** For any bounded sets $D,D_1,D_2,\ldots,D_n \subset X$, and $0 \le \tau \le s \le t \le T$,

$$\begin{aligned}
&\alpha(g(D)) \le \beta(t)\alpha(D),\\
&\alpha(\psi(t-s,s)f(s,D_1,D_2,\ldots,D_n)\\
\le\ &\beta_1(t,s)\alpha(D_1) + \beta_2(t,s)\alpha(D_2) + \cdots + \beta_n(t,s)\alpha(D_n),\\
&\alpha(\psi(t-s,s)\varphi(s,\tau)f(\tau,D_1,D_2,\ldots,D_n)\\
\le\ &\zeta_1(t,s,\tau)\alpha(D_1) + \zeta_2(t,s,\tau)\alpha(D_2) + \cdots + \zeta_n(t,s,\tau)\alpha(D_n),
\end{aligned}$$

where $\beta(t)$ is a nonnegative function, and $\sup_{t\in I}\beta(t) := \beta < \infty$,

$$\sup_{t\in I}\int_0^t \beta_i(t,s)ds := \beta_i < \infty, i = 1,2,\ldots,n,$$

$$\sup_{t\in I}\int_0^t\int_0^s \zeta_j(t,s,\tau)d\tau ds := \zeta_j < \infty, j = 1,2,\ldots,n.$$

**(B3)** $g : C(I;X) \to X$ is continuous and there exists

$$0 < \alpha_1 < (C + \widetilde{M(T)})^{-1},\quad \alpha_2 \ge 0$$

such that

$$\|g(u)\| \le \alpha_1\|u\| + \alpha_2.$$

**(B4)** The functions $\mu$ and $\omega$ satisfy the following condition:

$$C(1 + CB(q,\gamma))T^{\gamma}_{p,q},\Omega_{p,q}\Big(\sum_{i=1}^{n}K_i^*\Big)\|\mu\|_{L^p}\lim\inf_{\tau\to\infty}\frac{\omega(\tau)}{\tau} < 1 - \alpha_1(C + \widetilde{M(T)}),$$

where

$$\Omega_{p,q} = \Big(\frac{p-1}{pq-1}\Big)^{\frac{p-1}{p}},\ \text{and } T^{\gamma}_{p,q} = \max\{T_{p,q},T_{p,q+\gamma}\}.$$

**Theorem 3.1.** *Suppose that (B1) - (B4) are satisfied, and if* $(C+\widetilde{M(T)})\beta+4(\sum_{i=1}^{n}(\beta_i+2\zeta_i)K_i^*) < 1$, *then (1.1) has a mild solution on* $[0,T]$.

*Proof.* Define the operator $F : C(I;X) \to C(I;X)$ by

$$\begin{aligned} F(u)(t) &= A^{-1}(0)g(u) + u_0 + \int_0^t \psi(t-\eta,\eta)U(\eta)A(0)[A^{-1}(0)g(u) + u_0]d\eta \\ &+ \int_0^t \psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))d\eta \\ &+ \int_0^t \int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))dsd\eta \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)),\ t \in I. \end{aligned}$$

Then we proceed in five steps.

**Step 1.** We show that $F$ is continuous.

Let $u_i$ be a sequence such that $u_i \to u$ as $i \to \infty$. Since $f$ satisfies (B1) we have

$$f(t,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) \to f(t,(K_1u)(t),(K_2u)(t),\ldots,(K_nu)(t)),\ as\ i \to \infty. \tag{3.1}$$

Then

$$\begin{aligned} \|F(u_i)(t) - F(u)(t)\| &\le \|A^{-1}(0)\|\|g(u_i) - g(u)\| + \int_0^t \|\psi(t-\eta,\eta)U(\eta)\|\|g(u_i) - g(u)\|d\eta \\ &+ \int_0^t \|\psi(t-\eta,\eta)[f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \\ &- f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))]\|d\eta \\ &+ \int_0^t \int_0^\eta \|\psi(t-\eta,\eta)\varphi(\eta,s)[f(s,(K_1u_i)(s),(K_2u_i)(s),\ldots,(K_nu_i)(s)) \\ &- f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))]\|dsd\eta \\ &+ M_1m\rho\|u-u_i\|_\infty. \end{aligned}$$

According to the condition (A2), (2.3), and continuity of $g$, we have

$$\|A^{-1}(0)\|\|g(u_i) - g(u)\| \to 0, \quad as \quad i \to \infty;$$

$$\int_0^t \|\psi(t-\eta,\eta)U(\eta)\|\|g(u_i) - g(u)\|d\eta \to 0, \quad as \quad i \to \infty.$$

Noting that $u_i \to u$ in $C(I,X)$, there exists $\varepsilon > 0$ such that $\|u_i - u\| \le \varepsilon$ for $i$ sufficiently large. Therefore, we have

$$\begin{aligned} &\|[f(t,(K_1u_i)(t),(K_2u_i)(t),\ldots,(K_nu_i)(t)) - f(t,(K_1u)(t),(K_2u)(t),\ldots,(K_nu)(t))]\| \\ &\le \mu(t)\Big[\omega\Big(\sum_{j=1}^n \|K_ju_i)(t)\|\Big) + \omega\Big(\sum_{j=1}^n \|K_ju)(t)\|\Big)\Big] \\ &\le \mu(t)\Big[\omega\Big(\sum_{j=1}^n K_j^*(\|u\| + \varepsilon)\Big) + \omega\Big(\sum_{j=1}^n K_j^*\|u\|\Big)\Big]. \end{aligned}$$

Using (2.1) and by means of the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} &\int_0^t \|\psi(t-\eta,\eta)[f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \\ &- f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))]\|d\eta \\ &\le C\int_0^t (t-\eta)^{q-1}\|f(\eta,(K_1u_i)(\eta),(K_2u_i)(\eta),\ldots,(K_nu_i)(\eta)) \\ &- f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))]\|d\eta \to 0, \quad as \quad i \to \infty. \end{aligned}$$

Similarly, by (2.1) and (2.2), we have

$$\begin{aligned}&\int_0^t\int_0^\eta \|\psi(t-\eta,\eta)\varphi(\eta,s)[f(s,(K_1u_i)(t),(K_2u_i)(t),\dots,(K_nu_i)(t))\\&-f(s,(K_1u)(s),(K_2u)(s),\dots,(K_nu)(s))]\|dsd\eta\\&\le C^2\int_0^t\int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\gamma-1}\|f(s,(K_1u_i)(t),(K_2u_i)(t),\dots,(K_nu_i)(t))\\&-f(s,(K_1u)(s),(K_2u)(s),\dots,(K_nu)(s))\|dsd\eta\to 0,\quad as\quad i\to\infty.\end{aligned}$$

Therefore, we deduce that

$$\lim_{i\to\infty}\|F(u_i)-F(u)\|=0.$$

**Step 2.** We show that $F$ maps bounded sets of $C(I,X)$ into bounded sets in $C(I,X)$. For any $r>0$, we set $B_r=\{u\in C(I,X):\|u\|\le r\}$. Now, for $u\in B_r$, by (B1), we can see

$$\|f(t,(K_1u)(t),(K_2u)(t),\dots,(K_nu)(t))\|\le \mu(t)\omega\Big(\sum_{j=1}^n K_j^*r\Big). \tag{3.1}$$

Based on (2.3), we denote that $S(t):=\int_0^t\psi(t-\eta,\eta)U(\eta)d\eta$, we have

$$\|S(t)A(0)u_0\|\le C^2t^q\Big(\frac{1}{q}+t^\gamma B(q,\gamma+1)\Big)\|A(0)u_0\|=\widetilde{M(t)}\|A(0)u_0\|.$$

Then for any $u\in B_r$, by (A2), (2.1), (2.2), and Lemma 2.6, we have

$$\begin{aligned}\|(Fu)(t)\|\le&\ \|A^{-1}(0)g(u)\|+\|u_0\|+\|S(t)g(u)\|+\|S(t)A(0)u_0\|\\&+\int_0^t\|\psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\dots,(K_nu)(\eta))\|d\eta\\&+\int_0^t\int_0^\eta\|\psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\dots,(K_nu)(s))\|dsd\eta\\\le&\ (C+\widetilde{M(t)})\|g(u)\|+\|u_0\|+\widetilde{M(t)}\|A(0)u_0\|\\&+C\int_0^t(t-\eta)^{q-1}\mu(\eta)\omega\Big(\sum_{j=1}^nK_j^*r\Big)d\eta\\&+C^2\int_0^t\int_0^\eta(t-\eta)^{q-1}(\eta-s)^{\gamma-1}\mu(s)\omega\Big(\sum_{j=1}^nK_j^*r\Big)dsd\eta\\\le&\ \alpha_1(C+\widetilde{M(t)})\|u\|+\alpha_2(C+\widetilde{M(t)})+\|u_0\|+\widetilde{M(t)}\|A(0)u_0\|\\&+M_1\Big[C\int_0^t(t-\eta)^{q-1}\mu(\eta)d\eta+C^2B(q,\gamma)\int_0^t(t-\eta)^{q+\gamma-1}\mu(\eta)d\eta\Big]\\&+\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)),\ t\in I,\end{aligned}$$

where $M_1=\omega(\sum_{j=1}^nK_j^*r)$.

By means of the Hölder inequality, we have

$$\begin{aligned}&\int_0^t(t-\eta)^{q-1}\mu(\eta)d\eta=t^{\frac{pq-1}{p}}M_{p,q}\|\mu\|_{L^p}\le T_{p,q}\Omega_{p,q}\|\mu\|_{L^p},\\&\int_0^t(t-\eta)^{\gamma+q-1}\mu(\eta)d\eta\le T_{p,q}\Omega_{p,q+\gamma}\|\mu\|_{L^p}.\end{aligned}$$

Thus

$$\|(Fu)(t)\| \leq \alpha_1(C+\widetilde{M(T)})r+\alpha_2(C+\widetilde{M(T)})+\|u_0\|+\widetilde{M(T)}\|A(0)u_0\| \\ +M_1\Omega_{p,q}T_{p,q}^{\gamma}[C+C^2B(q,\gamma)]\|\mu\|_{L^p}+M_1m\rho_1 := \tilde{r}.$$

This means $F(B_r)\subset B_{\tilde{r}}$.

**Step 3.** We show that there exists $m\in N$ such that $F(B_m)\subset B_m$.

Suppose contrary that for every $m\in N$, there exists $u_m\in B_m$, for any $m<r$ and $t_m\in I$, such that $\|(Fu_m)(t_m)\|>m$. However, on the other hand

$$\|f(t,(K_1u_m)(t),(K_2u_m)(t),\ldots,(K_nu_m)(t))\| \leq \mu(t)\omega\Big(\sum_{j=1}^{n}K_j^*m\Big), \tag{3.2}$$

we have

$$\begin{aligned} m<\|(Fu_m)(t_m)\| &\leq \alpha_1(C+\widetilde{M(T)})\|u_m\|+\alpha_2(C+\widetilde{M(T)})+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\Big[C\int_0^{t_m}(t_m-\eta)^{q-1}\mu(\eta)d\eta+M_1m\rho_1 \\ &\quad+C^2B(q,\gamma)\int_0^{t_m}(t_m-\eta)^{q+\gamma-1}\mu(\eta)d\eta\Big] \\ &\leq \alpha_1(C+\widetilde{M(T)})\|u_m\|+\alpha_2(C+\widetilde{M(T)})+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\Omega_{p,q}T_{p,q}^{\gamma}[C+C^2B(q,\gamma)]\|\mu\|_{L^p}+M_1m\rho_1 \\ &\leq \alpha_1(C+\widetilde{M(T)})m+\alpha_2(C+\widetilde{M(T)})+\|u_0\| \\ &\quad+\widetilde{M(T)}\|A(0)u_0\|+M_1\Omega_{p,q}T_{p,q}^{\gamma}[C+C^2B(q,\gamma)]\|\mu\|_{L^p}+M_1m\rho_1. \end{aligned}$$

Dividing both sides by $m$ and taking the lower limit as $m\to\infty$, we obtain

$$C(1+CB(q,\gamma))T_{p,q}^{\gamma}\Omega_{p,q}\sum_{j=1}^{n}K_j^*\|\mu\|_{L^p}\liminf_{m\to\infty}\frac{w(m)}{m}+M_1m\rho_1 \geq 1-\alpha_1(C+\widetilde{M(T)});$$

which contradicts (B4).

**Step 4.** We now prove that $F$ is equicontinuous. Denote

$$F(u)(t)=A^{-1}(0)g(u)+u_0+\int_0^t\psi(t-\eta,\eta)U(\eta)A(0)[A^{-1}(0)g(u)+u_0]d\eta+G(u)(t)+M_1m\rho_1,$$

where

$$\begin{aligned} G(u)(t) &= \int_0^t\psi(t-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))d\eta \\ &+\int_0^t\int_0^{\eta}\psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))dsd\eta+\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)). \end{aligned}$$

We show that $G(u)(.)$ is equicontinuous.

Let $0<t_2<t_1<T$ and $u\in B_m$. Then

$$\|(Gu)(t_1)-(Gu)(t_2)\| \leq I_1+I_2+I_3+I_4+I_5,$$

where

$$
\begin{aligned}
I_1 &= \int_0^{t_2} \|[\psi(t_1-\eta,\eta)-\psi(t_2-\eta,\eta)]f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta,\\
I_2 &= \int_{t_2}^{t_1} \|\psi(t_1-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta,\\
I_3 &= \int_0^{t_2}\int_0^{\eta} \|[\psi(t_1-\eta,\eta)-\psi(t_2-\eta,\eta)]\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|dsd\eta,\\
I_4 &= \int_{t_2}^{t_1}\int_0^{\eta} \|\psi(t_1-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|dsd\eta,\\
I_5 &= M_1 m\rho\|t_1-t_2\|
\end{aligned}
$$

It follows from Lemma 2.8, (B1), and (3.2) that $I_1, I_3 \to 0$, as $t_2 \to t_1$.
For $I_2$, from (2.1),(3.2), and (B1), we have

$$
\begin{aligned}
I_2 &= \int_{t_2}^{t_1} \|\psi(t_1-\eta,\eta)f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta))\|d\eta\\
&\leq CM_1\int_{t_2}^{t_1}(t_1-\eta)^{q-1}\mu(\eta)d\eta \to 0,\ as\ t_2\to t_1.
\end{aligned}
$$

Similarly, by (2.1), (2.2), (B1) and Lemma (2.6), we have

$$
\begin{aligned}
I_4 &= \int_{t_2}^{t_1}\int_0^{\eta} \|\psi(t_1-\eta,\eta)\varphi(\eta,s)f(s,(K_1u)(s),(K_2u)(s),\ldots,(K_nu)(s))\|dsd\eta\\
&\leq C^2M_1\int_{t_2}^{t_1}(t_1-\eta)^{q-1}\int_0^{\eta}(\eta-s)^{\gamma-1}\mu(s)dsd\eta \to 0,\ as\ t_2\to t_1.
\end{aligned}
$$

**Step 5.** We show that $\alpha(F(H)) < \alpha(H)$ for every bounded set $H \subset B_m$. For any $\varepsilon > 0$, we can take a sequence $\{h_v\}_{v=1}^{\infty} \subset H$ such that

$$\alpha(H) \leq 2\alpha(\{h_v\}) + \varepsilon,$$

(cf. [4]). So it follows from Lemmas 2.14-2.4, 2.8, (2) in Remark 2.15, and (B2) that

$$
\begin{aligned}
\alpha(F(H)) &\leq C\alpha(g(H)) + \widetilde{M(T)}\alpha(g(H)) + 2\alpha(G\{h_v\}) + \varepsilon\\
&\leq C\alpha(g(H)) + \widetilde{M(T)}\alpha(g(H))\\
&\quad + 2\sup_{t\in I}\alpha\Big(\Big\{\int_0^t \psi(t-\eta,\eta)f(\eta,(K_1h_v)(\eta),(K_2h_v)(\eta),\ldots,(K_nh_v)(\eta))d\eta\Big\}\\
&\quad + \Big\{\int_0^t\int_0^{\eta}\psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1h_v)(s),(K_2h_v)(s),\ldots,(K_nh_v)(s))dsd\eta\Big\}\Big) + \varepsilon
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\beta\alpha(H)+\widetilde{M(T)}\beta\alpha(H)\\
&\quad+4\sup_{t\in I}\Big(\int_0^t\alpha(\{\psi(t-\eta,\eta)f(\eta,(K_1h_v)(\eta),(K_2h_v)(\eta),\ldots,(K_nh_v)(\eta))\})d\eta\Big)\\
&\quad+8\sup_{t\in I}\Big(\int_0^t\int_0^\eta\alpha(\{\psi(t-\eta,\eta)\varphi(\eta,s)f(s,(K_1h_v)(s),(K_2h_v)(s),\ldots,(K_nh_v)(s))\})dsd\eta\Big)+\varepsilon\\
&\leq C\beta\alpha(H)+\widetilde{M(T)}\beta\alpha(H)+4\sup_{t\in I}\Big(\int_0^t\Big(\sum_{i=1}^n\beta_i(t,\eta)K_i^*\Big)\alpha(\{h_v\})d\eta\Big)\\
&\quad+8\sup_{t\in I}\Big(\int_0^t\int_0^\eta\Big(\sum_{i=1}^n\zeta_i(t,\eta,s)K_i^*\Big)\alpha(\{h_v\})dsd\eta\Big)+\varepsilon\\
&\leq C\beta\alpha(H)+\widetilde{M(T)}\beta\alpha(H)+\Big(4\sum_{i=1}^n\beta_iK_i^*+8\sum_{i=1}^n\zeta_iK_i^*\Big)\alpha(\{h_v\})+\varepsilon\\
&=\Big[(C+\widetilde{M(T)})\beta+4\Big(\sum_{i=1}^n(\beta_i+2\zeta_i)K_i^*\Big)\Big]\alpha(H)+\varepsilon.
\end{aligned}
$$

Since $\varepsilon$ is arbitrary, we can obtain

$$\alpha(F(H))\leq\Big[(C+\widetilde{M(T)})\beta+4\Big(\sum_{i=1}^n(\beta_i+2\zeta_i)K_i^*\Big)\Big]\alpha(H)<\alpha(H).$$

In summary, we have proven that $F$ has a fixed point $\tilde{u}\in B_m$. Consequently, (1.1) has at least one mild solution.

□

Our next result is based on the Banach's fixed point theorem.

(G1) There exists a positive function $l(.)\in L^1(I,R^+)$ and a constant $\mu>0$ such that

$$\|g(u)-g(u^*)\|\leq\mu\|u-u^*\|,$$

and

$$\|f(t,v_1,v_2,\ldots,v_n)-f(t,w_1,w_2,\ldots,w_n)\|\leq l(t)\Big(\sum_{i=1}^n\|v_i-w_i\|\Big),(v_i-w_i)\in X^2,i=1,2,\ldots,n.$$

(G2) There exists a constant $0<\delta<1$ such that the function $\Lambda:I\to R^+$ defined by

$$\Lambda(t)=\mu(C+\widetilde{M(T)})+C\Big(\sum_{i=1}^nK_i^*\Big)\Gamma(q)I^ql(t)+C^2\Big(\sum_{i=1}^nK_i^*\Big)\Gamma(q)\Gamma(\gamma)I^{q+\gamma}l(t)+M_1m\rho\|u-u^*\|\leq\delta,\quad t\in I.$$

**Theorem 3.2.** *Assume that (G1), (G2) are satisfied, then (1.1) has a unique mild solution.*

*Proof.* Let $F$ be defined as in Theorem 3.1. For any $u,u^*\in C(I,X)$, we have

$$
\begin{aligned}
&\|f(t,(K_1u)(t),(K_2u)(t),\ldots,(K_nu)(t))-f(t,(K_1u^*)(t),(K_2u^*)(t)\ldots,(K_nu^*)(t))\|\\
&\leq l(t)\Big(\sum_{i=1}^n\|(K_iu)(t)-(K_iu^*)(t)\|\Big)\\
&\leq l(t)\sum_{i=1}^nK_i^*\|u-u^*\|.
\end{aligned}
$$

Thus, from (A2), (2.1), (2.2) and Lemma 2.6, we have

$$
\begin{aligned}
&\|(Fu)(t) - (Fu^*)(t)\| \\
&\leq \mu C\|u - u^*\| + \mu \int_0^t \|\psi(t-\eta,\eta)U(\eta)\|\|u - u^*\| d\eta \\
&+ \int_0^t \|\psi(t-\eta,\eta)\|\|f(\eta,(K_1u)(\eta),(K_2u)(\eta),\ldots,(K_nu)(\eta)) \\
&- f(\eta,(K_1u^*)(\eta),(K_2u^*)(\eta),\ldots,(K_nu^*)(\eta))\| d\eta \\
&+ \int_0^t \int_0^\eta \|\psi(t-\eta,\eta)\psi(\eta,s)\|\|f(s,(Ku)(s),(Hu)(s)) - f(s,(Ku^*)(s),(Hu^*)(s))\| ds d\eta + M_1 m\rho\|u - u^*\|_\infty \\
&\leq \|u - u^*\| \Big[\mu(C + \widetilde{M(T)}) + C\Big(\sum_{i=1}^n K_i^*\Big) \int_0^t (t-\eta)^{q-1} l(\eta) d\eta \\
&+ C^2\Big(\sum_{i=1}^n K_i^*\Big) \int_0^t \int_0^\eta (t-\eta)^{q-1}(\eta - s)^{\gamma-1} l(s) ds d\eta\Big] + M_1 m\rho\|u - u^*\|_\infty \\
&= \Big[\mu(C + \widetilde{M(T)}) + C\Big(\sum_{i=1}^n K_i^*\Big)\Gamma(q) I^q l(t) + C^2\Big(\sum_{i=1}^n K_i^*\Big)\Gamma(q)\Gamma(\gamma) I^{q+\gamma} l(t)\Big]\|u - u^*\| + M_1 m\rho\|u - u^*\|_\infty \\
&= \Lambda(t)\|u - u^*\| + M_1 m\rho\|u - u^*\|_\infty.
\end{aligned}
$$

We get

$$\|F(u) - F(u^*)\| \leq \delta\|u - u^*\|.$$

By the Banach contraction mapping principle, $F$ has a unique fixed point, which is a mild solution of the (1.1). □

## 4. Example

To illustrate the usefulness of our main result, we consider the following fractional differential equation:

$$
\begin{aligned}
\frac{\partial^q}{\partial t^q} u(t,\xi) &= b(t,\xi)\frac{\partial^2}{\partial \xi^2} u(t,\xi) + \frac{t^n}{n}\int_0^t (t-s)u(s,\xi)ds + \frac{t^n}{n}\int_0^t e^{-(t+s)} u(s,\xi)ds, \xi \in [0,1] \\
&\Delta y|_{t=\frac{1}{2}} = I_1(\frac{1^-}{2}), t \in J_1 := [0,1], t \neq \frac{1}{2}, \\
&u(t,0) = u(t,1) = 0 \\
&u(0,\xi) = -\int_0^\xi \int_0^y b^{-1}(0,x) \sin\Big|\frac{u}{\lambda}\Big| dx dy, \qquad (4.1)
\end{aligned}
$$

where $0 < q < 1,\ 0 \leq t \leq 1,\ \lambda > C + \widetilde{M(1)},\ n \in N,\ b(t,\xi)$ is continuous function and is uniformly Hölder continuous in $t$, i.e., there exists $C > 0$ and $\gamma \in (0,1)$ such that

$$\|b(t_1,\xi) - b(t_2,\xi)\| \leq C|t_1 - t_2|^\gamma, 0 \leq t_1 \leq t_2 \leq 1.$$

Let $X \in L^2([0,1],R)$ and define $A(t)$ by

$$
\begin{aligned}
&D(A(t)) = H^2(0,1) \cap H_0^1(0,1) = \{H^2(0,1) : z(0) = z(1) = 0\}, \\
&- A(t)(z) = b(t,\xi)z''.
\end{aligned}
$$

Then $-A(s)$ generates an analytic semigroup $\exp(-tA(s))$.
For $t \in [0,1], \xi \in [0,1]$, we set

$$\begin{aligned} u(t)(\xi) &= u(t,\xi), \\ g(u) &= \sin\left|\frac{u}{\lambda}\right|, \\ I_1(y) &= \frac{|y|}{3+|y|} \\ A^{-1}(0)g(u) &= -\int_0^\xi \int_0^y b^{-1}(0,x)\sin\left|\frac{u}{\lambda}\right| dxdy, \\ f(t,(K_1u)(t),(K_2u)(t)(\xi) &= \frac{t^n}{n}\int_0^t (t-s)u(s,\xi)ds + \frac{t^n}{n}\int_0^t e^{-(t+s)}u(s,\xi)ds, \end{aligned}$$

where

$$\begin{aligned} (K_1u(t))(\xi) &= \int_0^t (t-s)u(s,\xi)ds, \\ (K_2u(t))(\xi) &= \int_0^t e^{-(t+s)}u(s,\xi)ds, \end{aligned}$$

and

$$\begin{aligned} K_1^* = \sup_{t\in I}\int_0^t (t-s)ds &< \frac{1}{2} < \infty, \\ K_2^* = \sup_{t\in I}\int_0^t e^{-(t+s)}ds &= \frac{1}{4} < \infty, \\ |I_1(y_1)-I_1(y_2)| &\le \frac{1}{3}|x-y|. \end{aligned}$$

Moreover, we can get

$$\begin{aligned} \|g(u)\| &\le \frac{1}{\lambda}\|u\|, \\ \alpha(g(D)) &\le \frac{1}{\lambda}\alpha(D) \end{aligned}$$

for any $D \subset X$. Then the above equation (4.1) can be written in the abstract form as (1.1). On the other hand,

$$\begin{aligned} \|f(t,(Ku)(t),(Hu)(t)(\xi)\| &\le \frac{t^n}{n}(\|(K_1u)(t,\xi)\| + \|(K_2u)(t,\xi)\|) \\ &\le \frac{t^n}{n}(K_1^*\|u\| + K_2^*\|u\|) \\ &= \mu(t)\omega(K_1^*\|u\| + K_2^*\|u\|), \end{aligned}$$

where $\mu(t) = t^n, \omega(z) = \frac{z}{n}$ satisfying (B1). For any $u_1, u_2 \in X$,

$$\begin{aligned} &\|\psi(t-s,s)f(s,(K_1u_1)(s),(K_2u_1)(s))(\xi) - \psi(t-s,s)f(s,(K_1u_2)(s),(K_2u_2)(s))(\xi)\| \\ &\le \frac{Cs^n}{n}(t-s)^{q-1}(\|(K_1u_1)(s))(\xi) - (K_1u_2)(s))(\xi)\| + \|(K_2u_1)(s))(\xi) - (K_2u_2)(s))(\xi)\|). \end{aligned}$$

Therefore, for any bounded sets $D_1, D_2 \subset X$, we have

$$\alpha(\psi(t-s,s)f(s,D_1,D_2)) \le \frac{Cs^n}{n}(t-s)^{q-1}(\alpha(D_1)+\alpha(D_2)).$$

Moreover,

$$\frac{C}{n}\sup_{t\in[0,1]}\int_0^t (t-s)^{q-1}s^n ds = \frac{C}{n}\sup_{t\in[0,1]} t^{n+q}B(q,n+1) = \frac{C}{n}B(q,n+1) := \beta_1 = \beta_2.$$

Similarly, we obtain

$$\alpha(\psi(t-s,s)\varphi(s,\tau)f(\tau,D_1,D_2)) \le \frac{C^2}{n}(t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^n(\alpha(D_1)+\alpha(D_2)),$$

and

$$\frac{C^2}{n}\sup_{t\in[0,1]}\int_0^t\int_0^s (t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^n d\tau ds \le \frac{C^2}{n}B(q,\gamma)B(q+\gamma,n+1) := \zeta_1 = \zeta_2.$$

Suppose further that

$$(1)\quad \frac{3}{4n}C(1+CB(q,\gamma))\Big(\frac{p-1}{pq-1}\Big)^{\frac{p-1}{p}}\|\mu\|_{L^p} + \frac{1}{3} < 1 - \frac{C+\widetilde{M(1)}}{\lambda},$$

$$(2)\quad \frac{1}{\lambda}(C+\widetilde{M(1)}) + 3(\beta_1+2\zeta_1) + \frac{1}{3} < 1.$$

Then (4.1) has a mild solution by Theorem 3.1.

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