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Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent Bundle

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Abstract

The main purpose of the proposed paper is to study the tangent bundle of a SSNMC on statistical manifolds and its submanifolds. We investigate the relationship between the complete lifts of a statistical connection and SSNMC in statistical manifolds and its submanifolds and proposed and proved some theorems on it. We also proposed and proved some theorems regarding curvature tensor, Gauss, Codazzi and Ricci equations with respect to statistical connection and SSNMC to its tangent bundle.

Mathematics Subject Classification (2020): 58A30, 53B12, 53B25, 53C05, 53C07, 53C25

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1. Introduction

Friedmann and Schouten proposed the concept of a semi-symmetric connection on a differentiable manifold [1]. If a linear connection meets the expression $T(X_0, Y_0) = \eta_0(Y_0)X_0 - \eta_0(X_0)Y_0$ and is not torsion free, it is referred to as a semi-symmetric connection. The semi-symmetric metric connection (SSMC) is one that satisfies the semi-symmetric condition and is known to exist if $g = 0$, else it is non-metric.

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Later, Hayden and Yano refined this idea and attained a number of significant Riemannian manifold results ([2], [3]). Agashe and Chafle introduced the notion of SSNMC in 1992 [4] and based on their studied many researchers further developed by studying Riemannian manifold and its submanifolds and conducted numerous more studies to further expand this idea ([5]–[10]).

S. Amari's study of statistical inference problems in information geometry was the first to establish the concept of statistical structure in 1985 [11]. Every point on a differentiable manifold known as a statistical manifold reflects a probability distribution. A statistical manifold with infinite dimensions can be found in the collection of all probability measurements. Amari [11] proposed a new geometric method for the statistical model of discrete probability distribution. Information geometry applications for statistical manifolds include time series, convex analysis, linear and nonlinear systems, neural networks, linear programming, totally integrable dynamic systems, partial differential equations and geometric modelling [12]. An extension of a Riemannian structure with a Riemannian metric and its Levi-Civita connection is what the statistical manifold can be seen as. As a generalisation of Kähler structure, Kurose [13] established the concept of holomorphic statistical structure, while H. Furuhata investigated the concepts of Sasakian and Kenmotsu statistical structure, respectively ([14], [15]) and Bagher et al. [16] discovered certain statistical manifold curvature features. Recently Yildirim studied the semi-symmetric non-metric connection on statistical manifold [17].

Numerous geometers have explored the tangent bundle of differential geometry, including Yano and Kobayashi [18], Yano and Ishihara [19], Tani [20], Pandey and Chaturvedi [21]. The vertical, complete, and horizontal lifts of tensors as well as the connection from the manifold to its tangent bundle were developed by Yano and Ishihara [19]. On an almost Hermitian manifold, Kahler manifold, Kenmotsu manifold, Sasakian manifold, Riemannian manifold,and submanifold of Riemannian manifold, Khan studied tangent bundle immersed with Quarter-Symmetric, semi-symmetric non-metric, and semi-symmetric P- connections ([22]–[27). The tangent bundle of P-Sasakian manifolds with quarter symmetric metric connection (QSMC) was recently studied by Khan et al. [28].

We start this paper with introduction section and section 2 is devoted to preliminaries. In section 3 we obtain the lifts of the statistical manifolds to its tangent bundle . Section 4 and 5 relate with the investigation of the geometry of the lifts of SSNMC of statistical manifolds and the curvature tensor to its tangent bundle and some proposed theorems are also proved. In section 6 we investigate the lifts of the statistical submanifolds and we also prove that the complete lift of the induced connection on a submanifold is also a SSNMC in its tangent bundle and we also derive Gauss and Weingarten formulae that admit a SSNMC in the tangent bundle. Lastly the lift of the Gauss, Codazzi, and Ricci equations with respect to a statistical connection and SSNMC in the tangent bundle are obtained in the final section.

2. Preliminaries

Let M be a differentiable manifold and $T_0M = \bigcup_{p\in M} T_{0p}M$ be the tangent bundle, where $T_{0p}M$ is the tangent space at point $p \in M$ and $\pi : T_0M \to M$ is the natural bundle structure of T_0M over M. For any coordinate system (Q, x^h) in M, where (x^h) is local coordinate system in the neighborhood Q, then $(\pi^{-1}(Q), x^h, y^h)$ is coordinate system in T_0M, where (x^h, y^h) is induced coordinate system in $\pi^{-1}(Q)$ from (x^h) [19].

2.1 Vertical and complete lifts

Let f_0 be a function, X_0 a vector field, ω_0 a 1-form, a tensor field F_0 of type (1,1) and ∇_0 an affine connection in M. The vertical and complete lifts of a function f_0, a vector field X_0, a 1-form ω_0, a tensor field F_0 of type (1,1) and ∇_0 an affine connection are given by $f_0^v, X_0^v, \omega_0^v, F_0^v, \nabla_0^v$ and $f_0^c, X_0^c, \omega_0^c, F_0^c, \nabla_0^c$ respectively. The following formulas of complete and vertical lifts are defined by [19]

$$(f_0X_0)^v = f_0^vX_0^v,\ (f_0X_0)^c = f_0^cX_0^v + f_0^vX_0^c, \tag{2.1}$$

$$X_0^vf_0^v = 0,\ X_0^vf_0^c = X_0^cf_0^v = (X_0f_0)^v,\ X_0^cf_0^c = (X_0f_0)^c, \tag{2.2}$$

$$\omega_0(f_0^v) = 0,\ \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v,\ \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{2.3}$$

$$F_0^vX_0^c = (F_0X_0)^v,\ F_0^cX_0^c = (F_0X_0)^c, \tag{2.4}$$

$$[X_0,Y_0]^v = [X_0^c,Y_0^v] = [X_0^v,Y_0^c],\ [X_0,Y_0]^c = [X_0^c,Y_0^c], \tag{2.5}$$

$$\nabla^c{}_{X_0^c}Y_0^c = (\nabla_{X_0}Y_0)^c,\ \nabla^c{}_{X_0^c}Y_0^v = (\nabla_{X_0}Y_0)^v. \tag{2.6}$$

2.2 Statistical manifolds

In an n-dimensional Riemannian manifold $(\ddot{M}^n, g_0)$ with Riemannian metric g_0, we consider $\ddot{\nabla}$ be its affine connection and let $\widetilde{\nabla}$ be the Levi-Civita connection. The structure $(\ddot{M},\ddot{\nabla},g_0)$ is said to be a statistical manifold if $\ddot{\nabla}$ is an affine and torsion-free connection and $\ddot{\nabla}g_0$ satisfied the Codazzi equation

$$(\ddot{\nabla}_{X_0}g_0)(Y_0,Z_0) = (\ddot{\nabla}_{Y_0}g_0)(X_0,Z_0), \tag{2.7}$$

for all $X_0,Y_0,Z_0 \in \chi_0(\ddot{M})$, where $\chi_0(\ddot{M})$ is the set of all tangent vector fields on $\ddot{M}$. We know that there exists an affine connection $\ddot{\nabla}^*$ which is the dual of $\ddot{\nabla}$ with respect to g_0 such that

$$X_0g_0(Y_0,Z_0) = g_0(\ddot{\nabla}_{X_0},Z_0) + g_0(Y_0,\ddot{\nabla}^*_{X_0}Z_0). \tag{2.8}$$

Also the pair of connections $\ddot{\nabla}$ and $\ddot{\nabla}^*$ satisfies $(\ddot{\nabla}^*)^* = \ddot{\nabla}$, one can obtain

$$\widetilde{\nabla} = \frac{1}{2}(\ddot{\nabla} + \ddot{\nabla}^*). \tag{2.9}$$

The tensor field $\ddot{K}$ of type (1,2) on $(\ddot{M},\ddot{\nabla},g_0)$ is defined by

$$\ddot{K}_{X_0}Y_0 = \ddot{\nabla}_{X_0}Y_0 - \widetilde{\nabla}_{X_0}Y_0,\ \ddot{K}_{X_0}Y_0 = \frac{1}{2}(\ddot{\nabla}_{X_0}Y_0 - \ddot{\nabla}^*_{X_0}Y_0), \tag{2.10}$$

and $\ddot{K}$ is symmetric which gives

$$g_0(\ddot{K}_{X_0}Y_0,Z_0) = g_0(\ddot{K}_{X_0}Z_0,Y_0),\ \ddot{K}_{X_0}Y_0 = \ddot{K}_{Y_0}X_0. \tag{2.11}$$

The statistical curvature tensor field with respect to $\ddot{\nabla}$ is defined as

$$\ddot{R}(X_0,Y_0)Z_0 = \ddot{\nabla}_{X_0}\ddot{\nabla}_{Y_0}Z_0 - \ddot{\nabla}_{Y_0}\ddot{\nabla}_{X_0}Z_0 - \ddot{\nabla}_{[X_0,Y_0]}Z_0. \tag{2.12}$$

By replacing $\ddot{\nabla}$ with $\ddot{\nabla}^*$, we can obtain the statistical curvature tensor field $\ddot{R}^*$. The curvature tensor fields $\ddot{R}$ and $\ddot{R}^*$ satisfy

$$\ddot{R}(X_0,Y_0)Z_0 = -\ddot{R}(Y_0,X_0)Z_0,\ \ddot{R}^*(X_0,Y_0)Z_0 = -\ddot{R}^*(Y_0,X_0)Z_0, \tag{2.13}$$

$$\begin{gathered} g_0(\ddot{R}(X_0,Y_0)Z_0,W_0) = -g_0(\ddot{R}^*(X_0,Y_0)W_0,Z_0), \\ \ddot{R}(X_0,Y_0)Z_0 + \ddot{R}(Y_0,Z_0)X_0 + \ddot{R}(Z_0,X_0)Y_0 = 0. \end{gathered} \tag{2.14}$$

In a statistical manifolds $\ddot{M}$ let N be its submanifold with g_0 which is an induced metric. Also $\nabla^{\odot}$ and $h^{\odot}$ are the induced connection and the second form on N respectively. Then the Gauss and Weingarten formulas for the Levi-Civita connection are

$$\widetilde{\nabla}_{X_0} Y_0 = \nabla^{\odot}_{X_0} Y_0 + h^{\odot}(X_0, Y_0), \;\; \widetilde{\nabla}_{X_0} Q_0 = -A^{\odot}_{0Q_0} X_0 + D^{\odot}_{0X_0} Q_0, \tag{2.15}$$

for all $X_0, Y_0 \in \chi_0(N)$ and $Q_0 \in \chi_0(N^{\perp})$ in which $A^{\odot}_0$ is the shape operator and $D^{\odot}_0$ is the normal connection on $N^{\perp}$. According to the statistical connections on $\ddot{\nabla}$ and $\ddot{\nabla}^*$, the Gauss and Weingarten are given by [29]

$$\ddot{\nabla}_{X_0} Y_0 = \nabla_{X_0} Y_0 + h(X_0, Y_0), \;\; \ddot{\nabla}_{X_0} Q_0 = -A_{0Q_0} X_0 + D_{0X_0} Q_0, \tag{2.16}$$

$$\ddot{\nabla}^*_{X_0} Y_0 = \nabla^*_{X_0} Y_0 + h^*(X_0, Y_0), \;\; \ddot{\nabla}^*_{X_0} Q_0 = -A^*_{0Q_0} X_0 + D^*_{0X_0} Q_0, \tag{2.17}$$

for all $X_0, Y_0 \in \chi_0(N)$ and $Q_0 \in \chi_0(N^{\perp})$ in which ∇, ∇^* are statistical and dual connections on N [30]. A_0 and A^*_0 denotes the shape operators D_0 and D^*_0 are the normal connections on $N^{\perp}$ and h, h^* are second fundamental forms on N respectively. From 2.16 and 2.17 we get

$$g_0(A_{0Q_0} X_0, Y_0) = g_0(h(X_0, Y_0), Q_0), \;\; g_0(A^*_{0Q_0} X_0, Y_0) = g_0(h^*(X_0, Y_0), Q_0). \tag{2.18}$$

If the second fundamental forms vanish then N is regarded to be as totally geodesic submanifold with respect to $\ddot{\nabla}$ and $\ddot{\nabla}^*$. Furthermore, if

$$h(X_0, Y_0) = H_0 g_0(X_0, Y_0), \;\; h^*(X_0, Y_0) = H^*_0 g_0(X_0, Y_0), \tag{2.19}$$

then the submanifold is called totally umbilical submanifold, where H_0 and H^*_0 are the mean curvature vectors admitting $\ddot{\nabla}$ and $\ddot{\nabla}^*$.

3. Statistical manifolds in the tangent bundle

Let us denote T_0M to be the tangent bundle of a statistical manifold $\ddot{M}$. Taking complete lifts on equations (2.7–2.19), we obtain

$$(\ddot{\nabla}^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) = (\ddot{\nabla}^c_{Y^c_0} g^c_0)(X^c_0, Z^c_0), \tag{3.1}$$

$$X^c_0 g^c_0(Y^c_0, Z^c_0) = g^c_0(\ddot{\nabla}^c_{X^c_0} Y^c_0, Z^c_0) + g^c_0(Y^c_0, \ddot{\nabla}^{*c}_{X^c_0} Z^c_0), \tag{3.2}$$

$$\widetilde{\nabla}^c = \frac{1}{2}(\ddot{\nabla}^c + \nabla^{*c}), \tag{3.3}$$

$$\ddot{K}^c_{X^c_0} Y^c_0 = \ddot{\nabla}^c_{X^c_0} Y^c_0 - \widetilde{\nabla}^c_{X^c_0} Y^c_0, \;\; \ddot{K}^c_{X^c_0} Y^c_0 = \frac{1}{2}(\ddot{\nabla}^c_{X^c_0} Y^c_0 - \ddot{\nabla}^c_{X^{*c}_0} Y^c_0), \tag{3.4}$$

$$g^c_0(\ddot{K}^c_{X^c_0} Y^c_0, Z^c_0) = g^c_0(\ddot{K}^c_{X^c_0} Z^c_0, Y^c_0), \;\; \ddot{K}^c_{X^c_0} Y^c_0 = \ddot{K}^c_{Y^c_0} X^c_0, \tag{3.5}$$

$$\ddot{R}^c(X^c_0, Y^c_0) Z^c_0 = \ddot{\nabla}^c_{X^c_0} \ddot{\nabla}^c_{Y^c_0} Z^c_0 - \ddot{\nabla}^c_{Y^c_0} \ddot{\nabla}^c_{X^c_0} Z^c_0 - \ddot{\nabla}^c_{[X^c_0, Y^c_0]}, \tag{3.6}$$

$$\ddot{R}^c(X^c_0, Y^c_0) Z^c_0 = -\ddot{R}^c(Y^c_0, X^c_0) Z^c_0, \;\; \ddot{R}^{*c}(X^c_0, Y^c_0) Z^c_0 = -\ddot{R}^{*c}(Y^c_0, X^c_0) Z^c_0, \tag{3.7}$$

$$g_0^c(\ddot{R}^c(X_0^c,Y_0^c)Z_0^c,W_0^c)=-g_0^c(\ddot{R}^{*c}(X_0^c,Y_0^c)W_0^c,Z_0^c), \tag{3.8}$$

$$\ddot{R}^c(X_0^c,Y_0^c)Z_0^c+\ddot{R}^c(Y_0^c,Z_0^c)X_0^c+\ddot{R}^c(Z_0^c,X_0^c)Y_0^c=0, \tag{3.9}$$

$$\widetilde{\nabla}^c_{X_0^c}Y_0^c=\nabla^{\odot c}_{X_0^c}Y_0^c+h^{\odot c}(X_0^c,Y_0^c),\ \widetilde{\nabla}^c_{X_0^c}Q_0^c=-A^{\odot c}_{0Q_0^c}X_0^c+D^{\odot c}_{0X_0^c}Q_0^c, \tag{3.10}$$

$$\ddot{\nabla}^c_{X_0^c}Y_0^c=\nabla^c_{X_0^c}Y_0^c+h^c(X_0^c,Y_0^c),\ \ddot{\nabla}^c_{X_0^c}Q_0^c=-A^c_{0Q_0^c}X_0^c+D^c_{0X_0^c}Q_0^c, \tag{3.11}$$

$$\ddot{\nabla}^{*c}_{X_0^c}Y_0^c=\nabla^{*c}_{X_0^c}Y_0^c+h^{*c}(X_0^c,Y_0^c),\ \ddot{\nabla}^{*c}_{X_0^c}Q_0^c=-A^{*c}_{0Q_0^c}X_0^c+D^{*c}_{0X_0^c}Q_0^c, \tag{3.12}$$

$$g_0^c(A^c_{0Q_0^c}X_0^c,Y_0^c)=g_0^c(h^c(X_0^c,Y_0^c),Q_0^c), \tag{3.13}$$

$$g_0^c(A^{*c}_{0Q_0^c}X_0^c,Y_0^c)=g_0^c(h^{*c}(X_0^c,Y_0^c),Q_0^c), \tag{3.14}$$

$$h^c(X_0^c,Y_0^c)=H^c g_0^c(X_0^c,Y_0^c),\ h^{*c}(X_0^c,Y_0^c)=H^{*c}g_0^c(X_0^c,Y_0^c). \tag{3.15}$$

4. SSNMC of a statistical manifolds in the tangent bundle

In a statistical manifold $(\ddot{M},\ddot{\nabla},g_0)$, the linear connection $\breve{\nabla}$ on $\ddot{M}$ is given by [17]

$$\breve{\nabla}_{X_0}Y_0=\ddot{\nabla}_{X_0}Y_0+\eta_0(Y_0)X_0-\ddot{K}_{X_0}Y_0, \tag{4.1}$$

where η_0 is a 1-form associated with vector field ξ_0

$$\eta_0(X_0)=g_0(X_0,\xi_0), \tag{4.2}$$

for all $X_0\in\chi_0(\ddot{M})$. By using (2.10) in (4.1), we get

$$\breve{\nabla}_{X_0}Y_0=\ddot{\nabla}^*_{X_0}Y_0+\eta_0(Y_0)X_0+\ddot{K}_{X_0}Y_0. \tag{4.3}$$

The torsion tensor $\breve{T}$ with respect to $\breve{\nabla}$ is given by

$$\begin{aligned}\breve{T}(X_0,Y_0)\ &=\ddot{\nabla}_{X_0}Y_0+\eta_0(Y_0)X_0-\ddot{K}_{X_0}Y_0+\ddot{\nabla}^*_{X_0}Y_0+\eta_0(Y_0)X_0+\ddot{K}_{X_0}Y_0-[X_0,Y_0]\\ &=\eta_0(Y_0)X_0-\eta_0(X_0)Y_0.\end{aligned} \tag{4.4}$$

A linear connection satisfying (4.4) is called SSMC. For any vector fields X_0,Y_0,Z_0 on $\ddot{M}$, we have

$$(\breve{\nabla}_{X_0}g_0)(Y_0,Z_0)=X_0g_0(Y_0,Z_0)-g_0(\breve{\nabla}_{X_0}Y_0,Z_0)-g_0(Y_0,\breve{\nabla}_{X_0}Z_0). \tag{4.5}$$

Using (2.8),(2.11) and (4.1), we get

$$\begin{aligned}(\breve{\nabla}_{X_0}g_0)(Y_0,Z_0)\ &=g_0(\ddot{\nabla}_{X_0}Y_0,Z_0)+g_0(Y_0,\ddot{\nabla}^*_{X_0}Z_0)+g_0(Y_0,\ddot{\nabla}_{X_0}Z_0-2\ddot{K}_{X_0}Z_0)\\ &-g_0(\ddot{\nabla}_{X_0}Y_0+\eta_0(Y_0)X_0-\ddot{K}_{X_0}Y_0,Z_0)-g_0(Y_0,\ddot{\nabla}_{X_0}Z_0+\eta_0(Z_0)X_0\\ &-\ddot{K}_{X_0}Z_0)\\ &=-\eta_0(Y_0)g_0(X_0,Z_0)-\eta_0(Z_0)g_0(X_0,Y_0).\end{aligned} \tag{4.6}$$

So, $\breve{\nabla}g \neq 0$. Hence a linear connection $\breve{\nabla}$ defined by (4.1) satisfies (4.4) and (4.6) is called SSNMC. Let $(\ddot{M}^n, \ddot{\nabla}, g_0)$ be an n-dimensional statistical manifold and let TM_0 be its tangent bundle. Taking complete lifts of (4.1–4.6) we get

$$\breve{\nabla}^c_{X_0^c} Y_0^c = \ddot{\nabla}^c_{X_0^c} Y_0^c + \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c - \ddot{K}^c_{X_0^c} Y_0^c, \tag{4.7}$$

$$\eta_0^c(X_0^c) = g^c(X_0^c, \xi_0^c), \tag{4.8}$$

$$\breve{\nabla}^c_{X_0^c} Y_0^c = \ddot{\nabla}^{*c}_{X_0^c} Y_0^c + \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c + \ddot{K}^c_{X_0^c} Y_0^c, \tag{4.9}$$

$$\breve{T}^c(X_0^c, Y_0^c) = \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c - \eta_0^c(X_0^c)Y_0^v - \eta_0^v(X_0^c)Y_0^c, \tag{4.10}$$

$$\begin{aligned}(\breve{\nabla}^c_{X_0^c} g^c)(Y_0^c, Z_0^c) &= X_0^c g^c(Y_0^v, Z_0^c) + X_0^v g^c(Y_0^c, Z_0^c) \\ &-g^c(\breve{\nabla}^c_{X_0^c} Y_0^c, Z_0^c) - g^c(Y_0^c, \breve{\nabla}^c_{X_0^c} Z_0^c),\end{aligned} \tag{4.11}$$

$$\begin{aligned}(\breve{\nabla}^c_{X_0^c} g^c)(Y_0^c, Z_0^c) &= -\eta_0^c(Y_0^c)g^c(X_0^v, Z_0^c) - \eta_0^v(Y_0^c)g^c(X_0^c, Z_0^c) \\ &-\eta_0^c(Z_0^c)g^c(X_0^v, Y_0^c) - \eta_0^v(Z_0^c)g^c(X_0^c, Y_0^c).\end{aligned} \tag{4.12}$$

Equation (4.7) is said to be a SSNMC if the torsion tensor $\breve{T}^c$ of TM_0 with respect to $\breve{\nabla}^c$ satisfies (4.10) and the Riemannian metric g_0^c holds the relation (4.12). From now on, we will denote the statistical manifold $(\ddot{M}, \ddot{\nabla}, g_0)$ in the tangent bundle TM_0 as $\ddot{M}^{\ddot{\nabla}^c}$.

Theorem 4.1: *Let $\ddot{M}^{\ddot{\nabla}^c}$ be a statistical manifold admitting a SSNMC in the tangent bundle TM_0 which satisfies (4.10) and (4.12), then the SSNMC of the statistical manifold in the tangent bundle is given by (4.7) and satisfies (4.9).*

Proof. Let $\breve{\nabla}^c$ be the complete lifts of a linear connection defined on $\ddot{M}$ by

$$\breve{\nabla}^c_{X_0^c} Y_0^c = \ddot{\nabla}^c_{X_0^c} Y_0^c + \omega_0^c(X_0^c, Y_0^c), \tag{4.13}$$

where $\ddot{\nabla}^c$ is the complete lift of the statistical connection and W_0^c is complete lift of a tensor field of type $(0,2)$ defined on $\ddot{M}$ and we have $\breve{\nabla}^c$ satisfies (4.10) and (4.12) and from (4.12) and (4.13), we have

$$\begin{aligned}&-\eta_0^c(Y_0^c)g_0^c(X_0^v, Z_0^c) - \eta_0^v(Y_0^c)g_0^c(X_0^c, Z_0^c) - \eta_0^c(Z_0^c)g_0^c(X_0^v, Y_0^c) - \eta_0^v(Z_0^c)g_0^c(X_0^c, Y_0^c) \\ &= (\breve{\nabla}^c_{X_0^c} g_0^c)(Y_0^c, Z_0^c) \\ &= X_0^c g_0^c(Y_0^v, Z_0^c) + X_0^v g_0^c(Y_0^c, Z_0^c) - g_0^c(\breve{\nabla}^c_{X_0^c} Y_0^c, Z_0^c) - g_0^c(Y_0^c, \breve{\nabla}^c_{X_0^c} Z_0^c) \\ &= X_0^c g_0^c(Y_0^v, Z_0^c) + X_0^v g_0^c(Y_0^c, Z_0^c) - g_0^c(\ddot{\nabla}^c_{X_0^c} Y_0^c + \omega_0^c(X_0^c, Y_0^c), Z_0^c) \\ &-g_0^c(Y_0^c, \ddot{\nabla}^c_{X_0^c} Z_0^c + \omega_0^c(X_0^c, Z_0^c)).\end{aligned} \tag{4.14}$$

Then by considering (3.2), we have

$$\begin{aligned}&-\eta_0^c(Y_0^c)g_0^c(X_0^c, Z_0^c) - \eta_0^v(Y_0^c)g_0^c(X_0^c, Z_0^c) - \eta_0^c(Z_0^c)g_0^c(X_0^c, Y_0^c) - \eta_0^v(Z_0^c)g_0^c(X_0^c, Y_0^c) \\ &= g_0^c(\ddot{\nabla}^c_{X_0^c} Y_0^c, Z_0^c) + g_0^c(Y_0^c, \ddot{\nabla}^{*c}_{X_0^c} Z_0^c) - g_0^c(\ddot{\nabla}^c_{X_0^c} Y_0^c, Z_0^c) \\ &-g_0^c(W_0^c(X_0^c, Y_0^c), Z_0^c) - g_0^c(Y_0^c, \ddot{\nabla}^c_{X_0^c} Z_0^c) - g_0^c(Y_0^c, W_0^c(X_0^c, Z_0^c)) \\ &= -2g_0^c(Y_0^c, \ddot{K}^c_{X_0^c} Z_0^c) - g_0^c(W_0^c(X_0^c, Y_0^c), Z_0^c) - g_0^c(Y_0^c, W_0^c(X_0^c, Z_0^c)).\end{aligned} \tag{4.15}$$

Using (3.4) in (4.15), we have

$$\begin{aligned} g_0^c(W_0^c(X_0^c,Y_0^c),Z_0^c)+g_0^c(Y_0^c,W_0^c(X_0^c,Z_0^c)) &= \eta_0^c(Y_0^c)g_0^c(X_0^v,Z_0^c) \\ &+\eta_0^v(Y_0^c)g_0^c(X_0^c,Z_0^c)+\eta_0^c(Z_0^c)g_0^c(X_0^v,Y_0^c) \\ &+\eta_0^v(Z_0^c)g_0^c(X_0^c,Y_0^c)-2g_0^c(Y_0^c,\ddot{K}^c_{X_0^c}Z_0^c). \end{aligned} \tag{4.16}$$

Beside that from (4.13), we have

$$\breve{T}^c(X_0^c,Y_0^c)=W_0^c(X_0^c,Y_0^c)-W_0^c(Y_0^c,X_0^c), \tag{4.17}$$

From (4.16) and (4.17), we have

$$\begin{aligned} & g_0^c(\breve{T}^c(X_0^c,Y_0^c)Z_0^c)+g_0^c(\breve{T}^c(Z_0^c,X_0^c)Y_0^c)+g_0^c(\breve{T}^c(Z_0^c,Y_0^c)X_0^c) \\ &= 2g_0^c(W_0^c(X_0^c,Y_0^c),Z_0^c)-2\eta_0^c(Z_0^c)g_0^c(X_0^v,Y_0^c) \\ &-2\eta_0^v(Z_0^c)g_0^c(X_0^c,Y_0^c)+2g_0^c(Y_0^c,\ddot{K}^c_{X_0^C}Z_0^c). \end{aligned} \tag{4.18}$$

Also using (4.10) and the inner product properties, we obtain

$$\begin{aligned} & g_0^c(\breve{T}^c(X_0^c,Y_0^c)Z_0^c)+g_0^c(\breve{T}^c(Z_0^c,X_0^c)Y_0^c)+g_0^c(\breve{T}^c(Z_0^c,Y_0^c)X_0^c) \\ &= 2\eta_0^c(Y_0^c)g_0^c(X_0^v,Z_0^c)+2\eta_0^v(Y_0^c)g_0^c(X_0^c,Z_0^c) \\ &-2\eta_0^c(Z_0^c)g_0^c(X_0^v,Y_0^c)-2\eta_0^v(Z_0^c)g_0^c(X_0^c,Y_0^c), \end{aligned} \tag{4.19}$$

in which $g_0^c(\breve{T}^c(X_0^c,Y_0^c),Z_0^c)=\breve{T}^c(X_0^c,Y_0^c,Z_0^c)$. Then by equation (4.18) and (4.19), we get the result

$$W_0^c(X_0^c,Y_0^c)=\eta_0^c(Y_0^c)X_0^v+\eta_0^v(Y_0^c)X_0^c-\ddot{K}^c_{X_0^c}Y_0^c. \tag{4.20}$$

Further, if we consider (3.4) and (4.13), we can obtain (4.14).

5. Curvature tensor of statistical manifolds in the tangent bundle

Let us denote the curvature tensor associated with the SSNMC $\breve{\nabla}$ by $\breve{R}$. Similar to the definition of curvature tensor of a Riemannian manifold $\ddot{M}$ according to the Riemannian connection $\widetilde{\nabla}$, the curvature tensor of $\ddot{M}$ is defined according to SSNMC $\breve{\nabla}$ by

$$\breve{R}(X_0,Y_0)Z_0=\breve{\nabla}_{X_0}\breve{\nabla}_{Y_0}Z_0-\breve{\nabla}_{Y_0}\breve{\nabla}_{X_0}Z_0-\breve{\nabla}_{[X_0,Y_0]}Z_0, \tag{5.1}$$

for all $X_0,Y_0,Z_0\in\chi_0(\ddot{M})$. Taking the complete lifts of (5.1), we get

$$\breve{R}^c(X_0^c,Y_0^c)Z_0^c=\breve{\nabla}^c_{X_0^c}\breve{\nabla}^c_{Y_0^c}Z_0^c-\breve{\nabla}^c_{Y_0^c}\breve{\nabla}^c_{X_0^c}Z_0^c-\breve{\nabla}^c_{[X_0^c,Y_0^c]}Z_0^c. \tag{5.2}$$

Now, we obtain the relations between the complete lifts of the curvature tensor of $\breve{R}^c$, $\ddot{R}^c$ and $\breve{R}^c$, $\ddot{R}^{*c}$ with the statistical connection $\ddot{\nabla}^c,\ddot{\nabla}^{*c}$.

Theorem 5.1: *In a statistical manifold $\ddot{M}^{\ddot{\nabla}^c}$. The complete lifts of the curvature tensor $\breve{R}^c$ of a SSNMC $\breve{\nabla}^c$ in the tangent bundle TM_0 satisfies the conditions.*

$$\begin{aligned} \breve{R}^c(X_0^c,Y_0^c)Z_0^c &= \ddot{R}^c(X_0^c,Y_0^c)Z_0^c-(\ddot{\nabla}^c_{X_0^c}\ddot{K}^c)(Y_0^c,Z_0^c)+(\ddot{\nabla}^c_{Y_0^c}\ddot{K}^c)(X_0^c,Z_0^c)+\ddot{K}^c_{X_0^c}\ddot{K}^c(Y_0^c,Z_0^c) \\ &-\ddot{K}^c_{Y_0^c}\ddot{K}^c(X_0^c,Z_0^c)-g_0^c(\eta_0^c(X_0^c)\xi_0^v+\eta_0^v(X_0^c)\xi_0^c-\ddot{\nabla}^c_{X_0^c}\xi_0^c+\ddot{K}^c_{X_0^c}\xi_0^c,Z_0^c)Y_0^c \\ &+g_0^c(\eta_0^c(Y_0^c)\xi_0^v+\eta_0^v(Y_0^c)\xi_0^c-\ddot{\nabla}^c_{Y_0^c}\xi_0^c+\ddot{K}^c_{Y_0^c}\xi_0^c,Z_0^c)X_0^c, \end{aligned} \tag{5.3}$$

and

$$\breve{R}^c(X_0^c,Y_0^c)Z_0^c = \ddot{R}^{*c}(X_0^c,Y_0^c)Z_0^c + (\ddot{\nabla}^{*c}_{X_0^c}\ddot{K}^c)(Y_0^c,Z_0^c) \\ -(\ddot{\nabla}^{*c}_{Y_0^c}\ddot{K}^c)(X_0^c,Z_0^c) + \ddot{K}^c_{X_0^c}\ddot{K}^c(Y_0^c,Z_0^c) - \ddot{K}^c_{Y_0^c}\ddot{K}^c(X_0^c,Z_0^c) - g_0^c(\eta_0^c(X_0^c)\xi_0^v + \eta_0^v(X_0^c)\xi_0^c \\ -\ddot{\nabla}^{*c}_{X_0^c}\xi_0^c + \ddot{K}^c_{X_0^c}\xi_0^c, Z_0^c)Y_0^c + g_0^c(\eta_0^c(Y_0^c)\xi_0^v + \eta_0^v(Y_0^c)\xi_0^c - \ddot{\nabla}^{*c}_{Y_0^c}\xi_0^c + \ddot{K}^c_{Y_0^c}\xi_0^c, Z_0^c)X_0^c. \tag{5.4}$$

Proof. Using (3.2), (3.4), (3.6), (4.7) and (4.9), we will obtain the result.

Proposition 5.1: *In a statistical manifold $\ddot{M}^{\ddot{\nabla}^c}$. The following relation hold: [label=()]*

(i) $-g_0^c(\ddot{\nabla}^c_{X_0^c}\xi_0^c, W_0^c) + g_0^c(\ddot{K}^c_{X_0^c}\xi_0^c, W_0^c) = -g_0^c(\ddot{\nabla}^{*c}_{X_0^c}\xi_0^c, W_0^c) + g_0^c(\ddot{K}^c_{X_0^c}\xi_0^c, W_0^c)$

(ii) $g_0^c((\ddot{\nabla}^c_{Y_0^c}\ddot{K}^c)(X_0^c,Z_0^c),W_0^c) - g_0^c((\ddot{\nabla}^c_{X_0^c}\ddot{K}^c)(Y_0^c,Z_0^c),W_0^c)$
$= g_0^c((\ddot{\nabla}^{*c}_{Y_0^c}\ddot{K}^c)(X_0^c,W_0^c),Z_0^c) - g_0^c((\ddot{\nabla}^{*c}_{X_0^c}\ddot{K}^c)(Y_0^c,W_0^c),Z_0^c)$

(iii) $g_0^c(\ddot{K}^c_{X_0^c}\ddot{K}^c_{Y_0^c}Z_0^c, W_0^c) = g_0^c(\ddot{K}^c_{Y_0^c}\ddot{K}^c_{X_0^c}W_0^c, Z_0^c)$

Proof. Using (3.4), we get (i). For (ii), we use (3.5) and prove that

$$\begin{aligned} g_0^c(\ddot{K}^c_{\ddot{\nabla}^c_{Y_0^c}X_0^c}Z_0^c, W_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^c_{X_0^c}Y_0^c}Z_0^c, W_0^c) &= g_0^c(\ddot{K}^c_{[Y_0^c,X_0^c]}Z_0^c, W_0^c) \\ &= g_0^c(\ddot{K}^c_{[Y_0^c,X_0^c]}W_0^c, Z_0^c) \\ &= g_0^c(\ddot{K}^c_{\ddot{\nabla}^{*c}_{Y_0^c}X_0^c}W_0^c, Z_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^{*c}_{X_0^c}Y_0^c}W_0^c, Z_0^c). \end{aligned} \tag{5.5}$$

Now, using (3.2) and (3.5), we get

$$\begin{aligned} g_0^c((\ddot{\nabla}^c_{Y_0^c}\ddot{K}^c)(X_0^c,Z_0^c),W_0^c) &= g_0^c(\ddot{\nabla}^c_{Y_0^c}\ddot{K}^c_{X_0^c}Z_0^c, W_0^c) - g_0^c(\ddot{K}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c}Z_0^c, W_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^c_{Y_0^c}X_0^c}Z_0^c, W_0^c) \\ &= Y_0^c g_0^c(W_0^v, \ddot{K}^c_{X_0^c}Z_0^c) + Y_0^v g_0^c(W_0^c, \ddot{K}^c_{X_0^c}Z_0^c) - g_0^c(\ddot{K}^c_{X_0^c}Z_0^c, \ddot{\nabla}^{*c}_{Y_0^c}W_0^c) \\ &\quad -g_0^c(\ddot{K}^c_{X_0^c}W_0^c, \ddot{\nabla}^c_{Y_0^c}Z_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^c_{Y_0^c}X_0^c}Z_0^c, W_0^c) \\ &= Y_0^c g_0^c(W_0^v, \ddot{K}^c_{X_0^c}Z_0^c) + Y_0^v g_0^c(W_0^c, \ddot{K}^c_{X_0^c}Z_0^c) - g_0^c(\ddot{K}^c_{X_0^c}\ddot{\nabla}^{*c}_{Y_0^c}W_0^c, Z_0^c) \\ &\quad -Y_0^c g_0^c(Z_0^v, \ddot{K}^c_{X_0^c}W_0^c) - Y_0^v g_0^c(Z_0^c, \ddot{K}^c_{X_0^c}W_0^c) \\ &\quad +g_0^c(Z_0^c, \ddot{\nabla}^{*c}\ddot{K}^c_{X_0^c}W_0^c) - g^c(\ddot{K}^c_{\ddot{\nabla}^c_{Y_0^c}X_0^c}Z_0^c, W_0^c) \\ &= g^c(Z_0^c, \ddot{\nabla}^{*c}_{Y_0^c}\ddot{K}^c_{X_0^c}W_0^c) - g^c(\ddot{K}^c_{X_0^c}\ddot{\nabla}^{*c}_{Y_0^c}W_0^c, Z_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^c_{Y_0^c}X_0^c}Z_0^c, W_0^c). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we obtain

$$\begin{aligned} &g_0^c((\ddot{\nabla}^c_{Y_0^c}\ddot{K}^c)(X_0^c,Z_0^c),W_0^c) - g_0^c((\ddot{\nabla}^c_{X_0^c}\ddot{K}^c)(Y_0^c,Z_0^c),W_0^c) \\ &= g_0^c(Z_0^c, \ddot{\nabla}^{*c}_{Y_0^c}\ddot{K}^c_{X_0^c}W_0^c) - g_0^c(\ddot{K}^c_{X_0^c}\ddot{\nabla}^{*c}_{Y_0^c}W_0^c, Z_0^c) - g_0^c(\ddot{K}^c_{\ddot{\nabla}^{*c}_{Y_0^c}X_0^c}W_0^c, Z_0^c) \\ &\quad -g_0^c(Z_0^c, \ddot{\nabla}^{*c}_{X_0^c}\ddot{K}^c_{Y_0^c}W_0^c) + g_0^c(\ddot{K}^c_{Y_0^c}\ddot{\nabla}^{*c}_{X_0^c}W_0^c, Z_0^c) + g_0^c(\ddot{K}^c_{\ddot{\nabla}^{*c}_{X_0^c}Y_0^c}W_0^c, Z_0^c) \\ &= g_0^c((\ddot{\nabla}^{*c}_{Y_0^c}\ddot{K}^c)(X_0^c,W_0^c),Z_0^c) - g_0^c((\ddot{\nabla}^{*c}_{X_0^c}\ddot{K}^c)(Y_0^c,W_0^c),Z_0^c). \end{aligned} \tag{5.7}$$

For (iii), from symmetry property of $\ddot{K}^c$, we deduce

$$g_0^c(\ddot{K}^c_{X_0^c}\ddot{K}^c_{Y_0^c}Z_0^c,W_0^c)=g_0^c(\ddot{K}^c_{X_0^c}W_0^c,\ddot{K}^c_{Y_0^c}Z_0^c)=g_0^c(\ddot{K}^c_{Y_0^c}\ddot{K}^c_{X_0^c}W_0^c,Z_0^c). \tag{5.8}$$

Proposition 5.2: *In a statistical manifold $\ddot{M}^{\ddot{\nabla}^c}$ admitting a SSNMC $\breve{\nabla}^c$ in the tangent bundle TM_0. The complete lift of curvature tensor $\breve{R}^c$ with respect to $\breve{\nabla}^c$ in the tangent bundle satisfy*

(i) $\breve{R}^c(X_0^c,Y_0^c)Z_0^c=-\breve{R}^c(Y_0^c,X_0^c)Z_0^c.$

(ii)
$$\begin{aligned}
&\breve{R}^c(X_0^c,Y_0^c,Z_0^c,W_0^c)+\breve{R}^c(X_0^c,Y_0^c,W_0^c,Z_0^c)=\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)g_0^c(X_0^v,W_0^c)+\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g_0^c(X_0^c,W_0^c)\\
&+\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g_0^c(X_0^c,W_0^c)-g_0^c(Z_0^c,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^v,W_0^c)-g_0^c(Z_0^v,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^c,W_0^c)+g_0^c(Z_0^c,\ddot{K}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^v,W_0^c)\\
&+g_0^c(Z_0^v,\ddot{K}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^c,W_0^c)+\eta_0^c(Y_0^c)\eta_0^c(W_0^c)g_0^c(X_0^v,Z_0^c)+\eta_0^c(Y_0^c)\eta_0^v(W_0^c)g_0^c(X_0^c,Z_0^c)+\eta_0^v(Y_0^c)\eta_0^c(W_0^c)g_0^c(X_0^c,Z_0^c)\\
&-g_0^c(W_0^c,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^v,Z_0^c)-g_0^c(W_0^v,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^c,Z_0^c)+g_0^c(W_0^c,\ddot{K}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^v,Z_0^c)+g_0^c(W_0^v,\ddot{K}^c_{Y_0^c}\xi_0^c)g_0^c(X_0^c,Z_0^c)\\
&-\eta_0^c(X_0^c)\eta_0^c(Z_0^c)g_0^c(Y_0^v,W_0^c)-\eta_0^c(X_0^c)\eta_0^v(Z_0^c)g_0^c(Y_0^c,W_0^c)-\eta_0^v(X_0^c)\eta_0^c(Z_0^c)g_0^c(Y_0^c,W_0^c)+g_0^c(Z_0^c,\ddot{\nabla}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^v,W_0^c)\\
&+g_0^c(Z_0^v,\ddot{\nabla}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^c,W_0^c)-g_0^c(Z_0^c,\ddot{K}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^v,W_0^c)-g_0^c(Z_0^v,\ddot{K}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^c,W_0^c)+\eta_0^c(X_0^c)\eta_0^c(W_0^c)g_0^c(Y_0^v,Z_0^c)\\
&+\eta_0^c(X_0^c)\eta_0^v(W_0^c)g_0^c(Y_0^c,Z_0^c)+\eta_0^v(X_0^c)\eta_0^c(W_0^c)g_0^c(Y_0^c,Z_0^c)-g_0^c(W_0^c,\ddot{\nabla}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^v,Z_0^c)-g_0^c(W_0^v,\ddot{\nabla}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^c,Z_0^c)\\
&+g_0^c(W_0^c,\ddot{K}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^v,Z_0^c)+g_0^c(W_0^v,\ddot{K}^c_{X_0^c}\xi_0^c)g_0^c(Y_0^c,Z_0^c).
\end{aligned}$$

(iii)
$$\begin{aligned}
&\breve{R}^c(X_0^c,Y_0^c)Z_0^c+\breve{R}^c(Y_0^c,Z_0^c)X_0^c+\breve{R}^c(Z_0^c,X_0^c)Y_0^c=g_0^c(Z_0^c,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Y_0^v+g_0^c(Z_0^v,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Y_0^c\\
&-g_0^c(Z_0^c,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)X_0^v-g_0^c(Z_0^v,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)X_0^c+g_0^c(X_0^c,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)Z_0^v+g_0^c(X_0^v,\ddot{\nabla}^c_{Y_0^c}\xi_0^c)Z_0^c\\
&-g_0^c(X_0^c,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)Y_0^v-g_0^c(X_0^v,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)Y_0^c+g_0^c(Y_0^c,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)X_0^v+g_0^c(Y_0^v,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)X_0^c\\
&-g_0^c(Y_0^c,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Z_0^v-g_0^c(Y_0^v,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Z_0^c.
\end{aligned}$$

Proof. By considering (3.2), we get (i). Using (5.3), (5.4) and proposition 5.1, we get (ii). By cycling $\breve{R}^c$ on X_0^c,Y_0^c,Z_0^c and a direct computation we get (iii).

Corollary 5.1: *In a statistical manifold $(\ddot{M}^{\ddot{\nabla}^c})$ admitting a SSNMC $\breve{\nabla}^c$ in the tangent bundle TM_0, if η_0^c and η_0^v are closed, then*

$$\breve{R}^c(X_0^c,Y_0^c)Z_0^c+\breve{R}^c(Y_0^c,Z_0^c)X_0^c+\breve{R}^c(Z_0^c,X_0^c)Y_0^c=0.$$

Proof. By considering equation (3.2), we get

$$\begin{aligned}
&g_0^c(Z_0^c,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Y_0^v+g_0^c(Z_0^v,\ddot{\nabla}^c_{X_0^c}\xi_0^c)Y_0^c-g_0^c(X_0^c,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)Y_0^v-g_0^c(X_0^v,\ddot{\nabla}^c_{Z_0^c}\xi_0^c)Y_0^c\\
&=X_0^c g_0^c(\xi_0^c,Z_0^c)Y_0^v+X_0^c g_0^c(\xi_0^v,Z_0^c)Y_0^c+X_0^v g_0^c(\xi_0^c,Z_0^c)Y_0^c-g_0^c(\xi_0^c,\ddot{\nabla}^{*c}_{X_0^c}Z_0^c)Y_0^v\\
&\quad-g_0^c(\xi_0^v,\ddot{\nabla}^{*c}_{X_0^c}Z_0^c)Y_0^c-Z_0^c g_0^c(\xi_0^c,X_0^c)Y_0^v-Z_0^c g_0^c(\xi_0^v,X_0^c)Y_0^c-Z_0^v g_0^c(\xi_0^c,X_0^c)Y_0^c\\
&\quad+g_0^c(\xi_0^c,\ddot{\nabla}^{*c}_{Z_0^c}X_0^c)Y_0^v+g_0^c(\xi_0^v,\ddot{\nabla}^{*c}_{Z_0^c}X_0^c)Y_0^c\\
&=X_0^c\eta_0^v(Z_0^c)+X_0^v\eta_0^c(Z_0^c)-Z_0^c\eta_0^v(X_0^c)-Z_0^v\eta_0^c(X_0^c)-\eta_0^c[X_0^c,Z_0^c]\\
&=d\eta_0^c(X_0^c,Z_0^c)Y_0^v+d\eta_0^v(X_0^c,Z_0^c)Y_0^c.
\end{aligned} \tag{5.9}$$

Using (iii) of proposition 5.2, we have

$$\begin{aligned}
&\breve{R}^c(X_0^c,Y_0^c)Z_0^c+\breve{R}^c(Y_0^c,Z_0^c)X_0^c+\breve{R}^c(Z_0^c,X_0^c)Y_0^c\\
&=d\eta_0^c(X_0^c,Z_0^c)Y_0^v+d\eta_0^v(X_0^c,Z_0^c)Y_0^c+d\eta_0^c(Y_0^c,X_0^c)Z_0^v\\
&\quad+d\eta_0^v(Y_0^c,X_0^c)Z_0^c+d\eta_0^c(Z_0^c,Y_0^c)X_0^v+d\eta_0^v(Z_0^c,Y_0^c)X_0^c\\
&=0,
\end{aligned} \tag{5.10}$$

and this complete the proof.

6. SSNMC on statistical submanifolds in the tangent bundle

The Gauss formula admitting SSNMC $\breve{\nabla}$ is expressed as

$$\breve{\nabla}_{X_0}Y_0=\nabla^{\bullet}_{X_0}Y_0^c+h^{\bullet}(X_0,Y_0), \tag{6.1}$$

which $\nabla^{\bullet}$ and $h^{\bullet}$ denote the induced connection and second fundamental form on the submanifold N that admits a SSNMC $\breve{\nabla}$. If the second fundamental form $h^{\bullet}$ satisfies the condition $h^{\bullet}(X_0,Y_0)=H^{\bullet}g_0(X_0,Y_0)$. Then the submanifold N is said to be totally umbilical which admits a SSNMC $\breve{\nabla}$, where $H^{\bullet}$ is the mean curvature tensor with respect to $\breve{\nabla}$. Also if $h^{\bullet}$ vanishes, then N is called totally geodesic.

Let $\nabla^{\bullet c}$ and $h^{\bullet c}$ be the complete lifts of the induced connection and complete lifts of second fundamental form in the submanifold N of the tangent bundle TM_0 . Now taking a complete lifts of (6.1), we get

$$\breve{\nabla}^c_{X_0^c}Y_0^c=\nabla^{\bullet c}_{X_0^c}Y_0^c+h^{\bullet c}(X_0^c,Y_0^c). \tag{6.2}$$

Now we will examine the relationship between $\nabla^{\bullet c}$ and ∇^c in the tangent bundle TM_0, assuming $X_0^c,Y_0^c\in\xi_0(N)$.

Theorem 6.1: *In a statistical manifold $(\ddot{M}^{\ddot{\nabla}^c})$ with respect to SSNMC $\breve{\nabla}^c$ in the tangent bundle TM_0, N be a submanifold of $(\ddot{M}^{\ddot{\nabla}^c})$ and $X_0^c\in\xi_0(N)$. Then the following relations hold:*

$$\nabla^{\bullet c}_{X_0^c}Y_0^c=\nabla^c_{X_0^c}Y_0^c+\eta_0^c(Y_0^c)X_0^v+\eta_0^v(Y_0^c)X_0^c-K^c_{X_0^c}Y_0^c, \tag{6.3}$$

$$h^{\bullet c}(X_0^c,Y_0^c)=\frac{1}{2}(h^c(X_0^c,Y_0^c)+h^{*c}(X_0^c,Y_0^c)), \tag{6.4}$$

in which $K^c_{X_0^c}Y_0^c=\frac{1}{2}(\nabla^c-\nabla^{*c}),\forall X_0^c,Y_0^c\in\xi_0(N)$.

Proof. Using the Gauss formula and (4.7) in (3.12), we obtain the following

$$\begin{aligned}
\breve{\nabla}^c_{X_0^c}Y_0^c&=\ddot{\nabla}^c_{X_0^c}Y_0^c+\eta_0^c(Y_0^c)X_0^v+\eta_0^v(Y_0^c)X_0^c-\ddot{K}^c_{X_0^c}Y_0^c\\
&=\nabla^c_{X_0^c}Y_0^c+h^c(X_0^c,Y_0^c)+\eta_0^c(Y_0^c)X_0^v+\eta_0^v(Y_0^c)X_0^c\\
&\quad-\frac{1}{2}(\nabla^c_{X_0^c}Y_0^c+h^c(X_0^c,Y_0^c)-\nabla^{*c}_{X_0^c}Y_0^c-h^{*c}(X_0^c,Y_0^c))\\
&=\nabla^c_{X_0^c}Y_0^c+\eta_0^c(Y_0^c)X_0^v+\eta_0^v(Y_0^c)X_0^c-K^c_{X_0^c}Y_0^c\\
&\quad+\frac{1}{2}(h^c(X_0^c,Y_0^c)+h^{*c}(X_0^c,Y_0^c)).
\end{aligned} \tag{6.5}$$

If we separate the tangential and normal part we get our assertion.

Remark 6.1: *From the above proof, the relation of the complete lifts of* $\nabla^{\bullet c}$ *and* ∇^{*c} *in the tangent bundle is*

$$\nabla^{\bullet c}_{X_0^c} Y_0^c = \nabla^{*c}_{X_0^c} Y_0^c + \eta_0^c(Y_0^c) X_0^v + \eta_0^v(Y_0^c) X_0^c + \ddot{K}^c_{X_0^c} Y_0^c, \tag{6.6}$$

for all $X_0^c, Y_0^c \in \chi_0(N)$.

Now we can obtain the following corollaries by considering (3.3), (6.2), (6.3) and (6.4).

Corollary 6.1: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$ *and* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0. *Then the second fundamental form* $h^{\bullet c}$ *coincides with* $h^{\odot}$ *in the tangent bundle.*

Corollary 6.2: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$ *and* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0. *Then the induced connection* $\nabla^{\bullet c}$ *of the SSNMC* $\breve{\nabla}^c$ *is also SSNMCin the tangent bundle.*

Proposition 6.1: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$ *and* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0. *if* N *is totally umbilical with respect to the statistical connections then* N *is totally umbilical with respect to SSNMC in the tangent bundle.*

Proof. In view of (3.15), using (3.3) and (6.3), we obtain

$$\begin{aligned} h^{\bullet c}(X_0^c, Y_0^c) &= \frac{1}{2}(h^c(X_0^c, Y_0^c) + h^{*c}(X_0^c, Y_0^c)) \\ &= \frac{1}{2}(H^c + H^{*c}) g_0^c(X_0^c, Y_0^c). \end{aligned} \tag{6.7}$$

So, N is totally umbilical. Also, the mean curvature tensor that admits $\breve{\nabla}^c$ is

$$H^{\bullet c}(X_0^c, Y_0^c) = \frac{1}{2}(H^c + H^{*c}). \tag{6.8}$$

Theorem 6.2: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$ *and* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0 *and* $\xi_0^c \in \chi_0(N)$. *For all* $X_0, Y_0 \in \chi_0(M)$ *and* $Q_0^c \in \chi_0(N^{\perp})$

$$\breve{\nabla}^c_{X_0^c} Q_0^c = -\frac{1}{2}(A^c_{0Q_0^c} X_0^c + A^{*c}_{0Q_0^c} X_0^c) + \frac{1}{2}(D^c_{0X_0^c} Q_0^c + D^{*c}_{0X_0^c} Q_0^c) \tag{6.9}$$

Proof. Since $Q_0^c \in \chi_0(N^{\perp})$, and $\eta_0^c(Q_0^c) = 0$. Then equation (3.11) and (4.7) imply that

$$\begin{aligned} \breve{\nabla}^c_{X_0^c} Q_0^c &= \ddot{\nabla}^c_{X_0^c} Q_0^c + \eta_0^c(Q_0^c) X_0^v + \eta_0^v(Q_0^c) X_0^c - \ddot{K}^c_{X_0^c} Q_0^c \\ &= -A^c_{0Q_0^c} X_0^c + D^c_{0X_0^c} Q_0^c - \frac{1}{2}(\ddot{\nabla}^c_{X_0^c} Q_0^c - \ddot{\nabla}^{*c}_{X_0^c} Q_0^c) \\ &= -\frac{1}{2}(A^c_{0Q_0^c} X_0^c + A^{*c}_{0Q_0^c} X_0^c) + \frac{1}{2}(D^c_{0X_0^c} Q_0^c + D^{*c}_{0X_0^c} Q_0^c). \end{aligned} \tag{6.10}$$

Proposition 6.2: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$ *and* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0 *and* $\xi_0^c \in \chi_0(N)$. *Then* $A_0^{\bullet c}$ *and* $D_0^{\bullet c}$ *coincides with* $A^{\odot c}$ *and* $D^{\odot c}$ *in the tangent bundle respectively.*

Proof. The Weingarten formula admitting SSNMC is expressed as

$$\breve{\nabla}^c_{X^c_0} Q^c_0 = -A^{\bullet c}_{0Q^c_0} X^c_0 + D^{\bullet c}_{0X^c_0} Q^c_0. \tag{6.11}$$

If we consider the above equation with (6.5) and separating the tangential and normal parts, we have

$$A^{\bullet c}_{0Q^c_0} X^c_0 = \frac{1}{2}(A^c_{0Q^c_0} X^c_0 + A^{*c}_{0Q^c_0} X^c_0),\ D^{\bullet c}_{0X^c_0} Q^c_0 = \frac{1}{2}(D^c_{0X^c_0} Q^c_0 + D^{*c}_{0X^c_0} Q^c_0). \tag{6.12}$$

By considering (2.19) and (6.12), we obtain the results. In the light of (3.14), (6.4) and (6.12), we get

$$\begin{aligned} g^c_0(A^{\bullet c}_{0Q^c_0} X^c_0, Y^c_0) &= \frac{1}{2} g^c_0((A^c_{0Q^c_0} X^c_0 + A^{*c}_{0Q^c_0} X^c_0), Y^c_0) \\ &= \frac{1}{2} g^c_0((h^c(X^c_0, Y^c_0) + h^{*c}(X^c_0, y^c_0)), Q^c_0) \\ &= g^c_0(h^{\bullet c}(X^c_0, Y^c_0), Q^c_0). \end{aligned} \tag{6.13}$$

Theorem 6.3: *In a statistical manifolds $(\ddot{M}^{\ddot{\nabla}^c})$ with respect to SSNMC $\breve{\nabla}^c$, N be a submanifold of $\ddot{M}$ in the tangent bundle TM_0 and $\xi^c \in (N^\perp)$. then $\breve{\nabla}^c$ and $h^{\bullet c}$ coincides with ∇^c and $h^{\odot c}$ in the tangent bundle respectively.*

Proof. Let $\xi^c_0 \in \chi_0(N^\perp)$, then similar to the proof in theorem 6.1. If we obtain the Gauss formula as follows.

$$\breve{\nabla}^c_{X^c_0} Y^c_0 = \nabla^c_{X^c_0} Y^c_0 - K^c_{X^c_0} Y^c_0 + \frac{1}{2}(h^c(X^c_0, Y^c_0) + h^{*c}(X^c_0, Y^c_0)). \tag{6.14}$$

So, if we taking tangential part of above equation

$$\nabla^{\bullet c}_{Y^c_0} Y^c_0 = \nabla^c_{X^c_0} Y^c_0 - K^c_{X^c_0} Y^c_0. \tag{6.15}$$

Using (3.4) and (3.10) in (6.15), we get

$$\nabla^{\odot}_{X^c_0} Y^c_0 = \nabla^c_{X^c_0} Y^c_0 - K^c_{X^c_0} Y^c_0. \tag{6.16}$$

In that case, from the last equation we have $\nabla^{\bullet c} = \nabla^c$. then to show the coincides with of $h^{\bullet c}$ and $h^{\odot c}$, we take the normal part of (6.14), so

$$h^{\bullet c}(X^c_0, Y^c_0) = \frac{1}{2}(h^c(X^c_0, Y^c_0) + h^{*c}(X^c_0, Y^c_0)), \tag{6.17}$$

for any $X^c_0, Y^c_0 \in \chi_0(N)$. From equation (3.10),

$$h^{\bullet c}(X^c_0, Y^c_0) = h^{\odot c}(X^c_0, Y^c_0). \tag{6.18}$$

Thus, the result is clear.

Lemma 6.1: *In a statistical manifolds $(\ddot{M}^{\ddot{\nabla}^c})$ with respect to SSNMC $\breve{\nabla}^c$, N be a submanifold of $\ddot{M}$ in the tangent bundle TM_0. Then ∇^c is a non-metric connection in the tangent bundle.*

Proof. For any vector fields X^c_0, Y^c_0, Z^c_0 on $\ddot{M}$, we have

$$\begin{aligned} (\breve{\nabla}^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) &= X^c_0 g^c_0(Y^v_0, Z^c_0) + X^v_0 g^c(Y^c_0, Z^c_0) - g^c_0(\nabla^c_{X^c_0} Y^c_0, Z^c_0) \\ &\quad - g^c_0(Y^c_0, \breve{\nabla}^c_{X^c_0} Z^c_0). \end{aligned} \tag{6.19}$$

Using (3.2), (3.5) and (6.2) in (6.19), we get

$$\begin{aligned}(\breve{\nabla}^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) &= X^c_0 g^c_0(Y^v_0, Z^c_0) + X^v_0 g^c_0(Y^c_0, Z^c_0) - g^c_0(\nabla^c_{X^c_0} Y^c_0, Z^c_0) \\ &-\eta^c_0(Y^c_0) g^c_0(X^v_0, Z^c_0) - \eta^v_0(Y^c_0) g^c_0(X^c_0, Z^c_0) + g^c_0(K^c_{X^c_0} Y^c_0, Z^c_0) \\ &-g^c_0(Y^c_0, \nabla^c_{X^c_0} Z^c_0) - \eta^c_0(Z^c_0) g^c_0(X^v_0, Y^c_0) - \eta^v_0(Z^c_0) g^c_0(X^c_0, Y^c_0) \\ &+g^c_0(Y^c_0, K^c_{X^c_0} Z^c_0).\end{aligned} \tag{6.20}$$

Using (4.12) in (6.20), we have

$$\begin{aligned}&(\nabla^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) - \eta^c_0(Y^c_0) g^c_0(X^v_0, Z^c_0) - \eta^v_0(Y^c_0) g^c_0(X^c_0, Z^c_0) + g^c_0(K^c_{X^c_0} Y^c_0, Z^c_0) \\ &-\eta^c_0(Z^c_0) g^c_0(X^v_0, Y^c_0) - \eta^v_0(Z^c_0) g^c_0(X^c_0, Y^c_0) + g_0(Y^c_0, K^c_{X^c_0} Z^c_0) \\ &= -\eta^c_0(Y^c_0) g^c_0(X^v_0, Z^c_0) - \eta^v_0(Y^c_0) g^c_0(X^c_0, Z^c_0) - \eta^c_0(Z^c_0) g^c_0(X^v_0, Y^c_0) - \eta^v_0(Z^c_0) g^c_0(X^c_0, Y^c_0), \\ &\Rightarrow (\nabla^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) = -2 g^c_0(K^c_{X^c_0} Y^c_0, Z^c_0).\end{aligned} \tag{6.21}$$

Consequently, we obtain $\nabla^c g^c_0 \neq 0$, so ∇^c is a non-metric connection.

7. The Gauss, Codazzi and Ricci equation in the tangent bundle

In this section, the relationship between the lifts of Riemannian curvature tensors with respect to statistical connections and the SSNMC to the tangent bundle are examined. We denote the tangential and normal parts of the Riemannian curvature tensors $\breve{R}^c$ corresponding to the statistical connections by $R^{\bullet c}$ and $R^{\perp c}$ respectively. Using equations (6.2) and (6.11), we have

$$\begin{aligned}\breve{\nabla}^c_{X^c_0} \breve{\nabla}^c_{Y^c_0} Z^c_0 &= \nabla^{\bullet c}_{X^c_0} \nabla^{\bullet c}_{Y^c_0} Z^c_0 - A^{\bullet c}_{0h^{\bullet c}(Y^c_0, Z^c_0)} X^c_0 + D^{\bullet c}_{0X^c_0} h^{\bullet c}(Y^c_0, Z^c_0) \\ &+h^{\bullet c}(X^c_0, \nabla^{\bullet c}_{Y^c_0} Z^c_0).\end{aligned} \tag{7.1}$$

Interchanging X_0 and Y_0 in (7.1), we get $\breve{\nabla}^c_{Y^c_0} \breve{\nabla}^c_{X^c_0} Z^c_0$. Then

$$\begin{aligned}\breve{R}^c(X^c_0, Y^c_0) Z^c_0 &= R^{\bullet c}(X^c_0, Y^c_0) Z^c_0 + A^c_{0h^{\bullet c}(X^c_0, Z^c_0)} Y^c_0 - A^c_{0h^{\bullet c}(Y^c_0, Z^c_0)} X^c_0 \\ &+D^{\bullet c}_{0X^c_0} h^{\bullet c}(Y^c_0, Z^c_0) - D^{\bullet c}_{0Y^c_0} h^{\bullet c}(X^c_0, Z^c_0) + h^{\bullet c}(X^c_0, \nabla^{\bullet c}_{Y^c_0} Z^c_0) \\ &-h^{\bullet c}(Y^c_0, \nabla^{\bullet c}_{X^c_0} Z^c_0) - h^{\bullet c}([X^c_0, Y^c_0], Z^c_0).\end{aligned} \tag{7.2}$$

In the view of (7.2), taking $W_0 \in \chi_0(N)$, we get the Gauss equation admitting SSNMC expressed as

$$\begin{aligned}g^c_0(\ddot{R}^c(X^c_0, Y^c_0) Z^c_0, W^c_0) &= g^c_0(R^c(X^c_0, Y^c_0) Z^c_0, W^c_0) + g^c_0(h^{\bullet c}(X^c_0, Z^c_0), h^{\bullet c}(Y^c_0, W^c_0)) \\ &-g^c_0(h^{\bullet c}(X^c_0, W^c_0), h^{\bullet c}(Y^c_0, Z^c_0))\end{aligned} \tag{7.3}$$

Proposition 7.1: *In a statistical manifold* $(\ddot{M}^{\breve{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$, N *a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0 *and* $\xi^c_0 \in \chi_0(N)$. *Then, we have*

$$
\begin{aligned}
g_0^c(\breve{R}^c(X_0^c,Y_0^c)Z_0^c,W_0^c) &= g_0^c(R^c(X_0^c,Y_0^c)Z_0^c,W_0^c) + g_0^c((\nabla^c_{Y_0^c}K^c)(X_0^c,Z_0^c)W_0^c) \\
&+g_0^c(Z_0^c,\nabla^c_{X_0^c}\xi_0^c)g_0^c(Y_0^v,W_0^c) + g_0^c(Z_0^v,\nabla^c_{X_0^c}\xi_0^c)g_0^c(Y_0^c,W_0^c) \\
&-\eta_0^c(K^c_{X_0^c}Z_0^c)g_0^c(Y_0^v,W_0^c) - \eta_0^v(K^c_{X_0^c}Z_0^c)g_0^c(Y_0^c,W_0^c) \\
&+\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)g_0^c(X_0^v,W_0^c) + \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g_0^c(X_0^c,W_0^c) \\
&+\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g_0^c(X_0^c,W_0^c) - g_0^c((\nabla^c_{X_0^c}K^c)(Y_0^c,Z_0^c),W_0^c) \\
&-\eta_0^c(K^c_{Y_0^c}Z_0^c)g_0^c(X_0^v,W_0^c) - \eta_0^v(K^c_{Y_0^c}Z_0^c)g_0^c(X_0^c,W_0^c) \\
&+g_0^c(K^c(X_0^c,(Y_0^c,Z_0^c),W_0^c)) - g_0^c(Z_0^c,\nabla^c_{Y_0^c}\xi_0^c)g_0^c(X_0^v,W_0^c) \\
&-g_0^c(Z_0^v,\nabla^c_{Y_0^c}\xi_0^c)g_0^c(X_0^c,W_0^c) - \eta_0^c(X_0^c)\eta_0^c(Z_0^c)g_0^c(Y_0^v,W_0^c) \\
&-\eta_0^c(X_0^c)\eta_0^v(Z_0^c)g_0^c(Y_0^c,W_0^c) - \eta_0^v(X_0^c)\eta_0^c(Z_0^c)g_0^c(Y_0^c,W_0^c) \\
&-g_0^c(K^c(Y_0^c,K^c(X_0^c,Z_0^c),W_0^c)) + \frac{1}{4}g_0^c(h^c(X_0^c,Z_0^c) + h^{*c}(X_0^c,Z_0^c), \\
&h^c(Y_0^c,W_0^c) + h^{*c}(Y_0^c,W_0^c)) - \frac{1}{4}g_0^c(h^c(X_0^c,W_0^c) + h^{*c}(X_0^c,W_0^c), \\
&h^c(Y_0^c,Z_0^c) + h^{*c}(Y_0^c,Z_0^c)),
\end{aligned}
$$

in which R^c is the curvature tensor of the induced statistical connection ∇^c on N in the tangent bundle.

Proof. The proof can be obtained from equations (6.3), (6.4) and (7.3).

Theorem 7.1: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$*, let* N *be a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0 *and* $\xi_0 \in \chi_0(N)$*. Then, the Codazzi equation with respect to SSNMC* $\breve{\nabla}^c$ *is given by*

$$
\begin{aligned}
g_0^c(\breve{R}^c(X_0^c,Y_0^c)Z_0^c,\lambda^c) &= g_0^c((\nabla^{\bullet c}_{X_0^c}h^{\bullet c})(Y_0^c,Z_0^c),\lambda^c) - g_0^c((\nabla^{\bullet c}_{Y_0^c}h^{\bullet c})(X_0^c,Z_0^c),\lambda^c) \\
&+\eta_0^c(Y_0^c)g_0^c(\lambda^v,h^{\bullet c}(X_0^c,Z_0^c)) + \eta_0^v(Y_0^c)g_0^c(\lambda^c,h^{\bullet c}(X_0^c,Z_0^c)) \\
&-\eta_0^c(X_0^c)g_0^c(\lambda^v,h^{\bullet c}(Y_0^c,Z_0^c)) - \eta_0^v(X_0^c)g_0^c(\lambda^c,h^{\bullet c}(Y_0^c,Z_0^c)),
\end{aligned}
\tag{7.4}
$$

in which

$$
(\nabla^{\bullet c}_{X_0^c}h^{\bullet c})(Y_0^c,Z_0^c) = D^{\bullet c}_{X_0^c}h^{\bullet c}(Y_0^c,Z_0^c) - h^{\bullet c}(\nabla^{\bullet c}_{X_0^c}Y_0^c,Z_0^c) - h^{\bullet c}(Y_0^c,\nabla^{\bullet c}_{X_0^c}Z_0^c).
$$

Proof. Since $\lambda^c \in \chi_0(N^\perp)$ and the inner product of (7.2) implies

$$
\begin{aligned}
g_0^c(\breve{R}^c(X_0^c,Y_0^c)Z_0^c,\lambda^c) &= g_0^c(h^{\bullet c}(X_0^c,\nabla^{\bullet c}_{Y_0^c}Z_0^c),\lambda^c) + g_0^c(D^{\bullet c}_{X_0^c}h^{\bullet c}(Y_0^c,Z_0^c),\lambda^c) \\
&-g_0^c(h^{\bullet c}(Y_0^c,\nabla^{\bullet c}_{X_0^c}Z_0^c),\lambda^c) + g_0^c(D^{\bullet c}_{Y_0^c}h^{\bullet c}(X_0^c,Z_0^c),\lambda^c) \\
&-g_0^c(h^{\bullet c}([X_0^c,Y_0^c],Z_0^c),\lambda^c).
\end{aligned}
\tag{7.5}
$$

If the equal of the lie bracket of X_0^c and Y_0^c are taken in equation (7.5), then the proof is completed.

Theorem 7.2: *In a statistical manifold* $(\ddot{M}^{\ddot{\nabla}^c})$ *with respect to SSNMC* $\breve{\nabla}^c$*,* N *a submanifold of* $\ddot{M}$ *in the tangent bundle* TM_0 *and* $\xi_0 \in \chi_0(N)$*. Then the Ricci equation with respect to SSNMC* $\breve{\nabla}^c$ *in the tangent bundle is given by*

$$g_0^c(\breve{R}^c(X_0^c,Y_0^c)Q_0^c,P_0^c)=g_0^c(R^{\perp c}(X_0^c,Y_0^c)Q_0^c,P_0^c)+g_0^c([A^{\bullet c}_{Q_0^c},A^{\bullet c}_{P_0^c}]X_0^c,Y_0^c),$$

where $X_0^c,Y_0^c\in\chi_0(N),Q_0,P_0\in\chi_0(N^{\perp})$ and $[A^{\bullet c}_{Q_0^c},A^{\bullet c}_{P_0^c}]=A^{\bullet c}_{Q_0^c}A^{\bullet c}_{P_0^c}-A^{\bullet c}_{P_0^c}A^{\bullet c}_{Q_0^c}$.

Proof. In view of equations (6.2) and (6.11), we obtain

$$\begin{aligned}g_0^c(\breve{R}^c(X_0^c,Y_0^c)Q_0^c,P_0^c) &= g_0^c(\breve{\nabla}^c_{X_0^c}\breve{\nabla}^c_{Y_0^c}Q_0^c,P_0^c)-g_0^c(\breve{\nabla}^c_{Y_0^c}\breve{\nabla}^c_{X_0^c}Q_0^c,P_0^c)\\ &-g_0^c(\breve{\nabla}^c_{[X_0^c,Y_0^c]}Q_0^c,P_0^c)\\ &=g_0^c(R^{\perp c}(X_0^c,Y_0^c)Q_0^c,P_0^c)-g_0^c(h^{\bullet c}(X_0^c,A^{\bullet c}_{Q_0^c}Y_0^c),P_0^c)\\ &+g_0^c(Y_0^c,A^{\bullet c}_{Q_0^c}X_0^c,P_0^c),\end{aligned}\tag{7.6}$$

for any $X_0^c,Y_0^c\in\chi_0(N)$ and $Q_0^c,P_0^c\in\chi_0(N^{\perp})$. Then by considering equation (6.13) in (7.6), we obtain

$$\begin{aligned}g_0^c(\breve{R}^c(X_0^c,Y_0^c)Q_0^c,P_0^c) &= g_0^c(R^{\perp c}(X_0^c,Y_0^c)Q_0^c,P_0^c)+g_0^c(A^{\bullet c}_{Q_0^c}A^{\bullet c}_{P_0^c}X_0^c,Y_0^c)\\ &-g_0^c(A^{\bullet c}_{P_0^c}A^{\bullet c}_{Q_0^c}Y_0^c,X_0^c)\\ &=g_0^c(R^{\perp c}(X_0^c,Y_0^c)Q_0^c,P_0^c)+g_0^c([A^{\bullet c}_{Q_0^c},A^{\bullet c}_{P_0^c}]X_0^c,Y_0^c).\end{aligned}$$

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References

[1] Friedman, A., Schouten, J. A., *Uber die Geometrie der halbsymmetrischen Ubertragungen*, Math. Zeitschr 21, (1924), 211–223.
[2] Hayden, H. A., *Sub-spaces of a space with torsion*, Proc., Lond. Math. Soc. 34, (1932), 27–50.
[3] Yano, K., *On semi-symmetric metric connections*, Rev. Roum. Math. Pures Appl. 15, (1970), 1579–1586.
[4] Agashe, N. S., *A semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. 23(6), (1992), 399–409.
[5] Aktan, N., *On non-existence of lightlike hypersurfaces of indefinite Kenmotsu space form*, Turk. J. Math. 32, (2008), 127–139.
[6] Beyendi, S., Aktan, N., Sivridag, A. I., *Almost α-cosymplectic f-manifolds endowed with a semi-symmetric non-metric connection connection*, Honam Math. J. 42(1), (2020), 175–185.
[7] Beyendi, S., Aktan, N., Sivridag, A. I., *On semi-invariant submanifolds of almost α-cosymplectic f-manifolds admitting a semi-symmetric non-metric connection connection,* Palest. J. Math. 9(2), (2020), 801–810.
[8] Sengupta, J., De, U. C., Binh, T. Q., *On a type of semi-symmetric non-metric connection on a Riemannnian manifold*, indian J. Pure Appl. Math. 31(12), (2000), 1659–1670.
[9] Yoldaş, H. İ., Haseeb A., Mofarreh, F., *Certain Curvature Conditions on Kenmotsu Manifolds and *-η-Ricci Solitons*, Axioms 12(2), (2023), 140.
[10] Yoldaş, H. İ., Meriç S. E., Yaşar, E., *On generic submanifold of Sasakian manifold with concurrent vector field*, Commun. Fac. of Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2), (2019), 1983–1994.
[11] Amari, S., *Differential-Geometrical Methods in Statistics*, Springer-Verlag, New York, (1985).
[12] Abdel-All, N. H., Abd-Ellah, H. N., Moustafa, H. M., *Information geometry and statistical manifold*, Chaos Solitons Fractals, 15(1), (2003), 161–172.
[13] Kurose, T., *Geometry of statistical manifolds in Mathematics in the 21st Century*, Nihon-Hyouron-Sha, Japan, (2004).
[14] Furuhata, H., *Hypersurface in statistical manifolds*, Differ. Geom. Appl. 27(3), (2009), 420–429.
[15] Furuhata, H., Hasegawa, I., Okuyama Y., Sato, K., *Kenmotsu statistical manifolds and warped product,* J. Geom. 108(3), (2017), 1175–1191.
[16] Bagher, M., Balgeshir, K., Salahvarzi, S., *Curvature of semi-symmetric metric connections on statistical manifolds*, Commun. Korean Math. Soc. 36(1), (2021), 149–164.

[17] Yildirim, M., *Semi-symmetric non-metric connections on statistical manifolds*, J. Geom. and Physics. 176, (2022), 104505.

[18] Yano, K., Kobayashi, S., *Prolongations of tensor fields and connections to tangent bundles I general theory*, Journal of the Mathematical Society of Japan 18(2), (1966), 194–210.

[19] Yano, K., Ishihara, S., *Tangent and cotangent bundles: differential geometry*, Marcel Dekker, Inc.: New York, NY, USA, (1973).

[20] Tani, M., *Prolongations of hypersurfaces to tangent bundles*, Kodai Mathematical Seminar Reports 21(1), (1969), 85–96.

[21] Pandey, P., Chaturvedi, B. B., *On a Kaehler manifold equipped with lift of quarter symmetric non-metric connection*, Facta Universitatis, Series: Mathematics and Informatics 33(4), (2018), 539–546.

[22] Khan, M. N. I., *Lift of semi-symmetric non-metric connection on a Kähler manifold*, Afrika Matematika 27, (2016), 345–352.

[23] Khan, M. N. I., *Tangent bundles endowed with semi-symmetric non-metric connection on a Riemannian manifold*, Facta Universitatis, Series: Mathematics and Informatics 36(4), (2021), 855–878.

[24] Khan, M. N. I., *Complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle*, International Journal of Mathematics and Computer Science 17(2), (2022), 909–916.

[25] Khan, M. N. I., *Submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle*, International Journal of Mathematics and Computer Science 17(1), (2022), 265–275.

[26] Khan, M. N. I., *Liftings from a para-sasakian manifold to its tangent bundles*, Filomat 37(20), (2023), 6727–6740.

[27] Khan, M. N. I., De, U. C., Velimirovic, L. S., *Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle*, Mathematics 11(1), (2023), 53.

[28] Khan, M. N. I., Mofarreh, F., Haseeb, A., *Tangent bundles of P-Sasakian manifolds endowed with a quarter-symmetric metric connection*, Symmetry 15(3), (2023), 753.

[29] Aydin, M. E., Mihai, A., Mihai, I., *Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature*, Bull. Math. Sci. 7, (2017), 155–166.

[30] Calin, O., Udriste, C., *Geometric modelling in Probability and Statistics*, Springer, (2014).