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Investigation of near fixed points, near fixed interval ellipse and its equivalence classes

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Abstract

The objective of the manuscript is to employ the Hardy-Roger contraction to determine the near fixed point and its unique equivalence class in the context of the $b-$interval metric space. Further, an improved $b-$interval metric variant of a quasi-contraction characterizing the completeness of a $b-$interval metric space is exhibited. Various illustrations have been provided to show the existence of a near fixed point and its distinct equivalence class for both continuous and discontinuous maps developed in the $b-$interval metric space. As an application of the $b-$interval metric, a near-fixed interval ellipse and its unique equivalence $\mathcal{E}-$class are introduced to study the geometry of non-unique near-fixed points.

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1. Introduction

At first, the idea of distance showed up during the period of Euclid. However, it is one of the earliest perceptions appreciated by humans. In 1906, Maurice Rene Frećhet [9] presented the well-known and

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acceptable form of a distance and named it “L-space". Felix Hausdorff [11] looked into it as a metric space. Subsequently, various refined, generalized, and extended versions of the metric structure showed up in the literature. For insights regarding the generalizations of the metric space, allude to Kirk and Shahzad [20]. Banach’s result [1] has been reported in the majority of these generalizations and extensions.

In recent years, particularly compelling extensions have emerged in the study of near-fixed points, where exact fixed points may not exist, but mappings remain “close” to identity in a generalized metric sense. Examples include works by Khojasteh et al. [19], Roldán López de Hierro et al. [27], and Ullah et al. [34], who all built near-fixed or near-coincidence results using simulation functions, $\alpha-$admissibility, and other contractive conditions. In 2018, Hsien-Chung Wu [35] established and popularized the notion of a metric interval space. Wu [35] familiarized metric interval spaces by exploiting the null set to study near fixed points. It is interesting to mention that metric interval space is not a conventional metric space, and all the closed and bounded intervals on the collection of real numbers may not be a vector space, as the additive inverse of each of its elements may not exist in it. A particularly relevant contribution is by Sarwar et al. [29], who established near-fixed-point theorems in metric interval and normed interval spaces via a novel $Z-$contraction framework. Their results beautifully tie traditional fixed-point theory to interval-valued mappings. In order to generalize and extend metric interval space, Joshi and Anita Tomar [14] proposed a unique distance structure known as a $b-$interval metric space. In addition, they describe topological concepts such as an open ball, closed ball, $b-$convergence, $b-$Cauchy sequence, and completeness of the space on a $b-$interval metric space with the goal to establish a distinct equivalence class of near fixed points and a setting in which a near fixed point will exist. For work on a near fixed point, near fixed interval circle, and near fixed interval disc in metric interval space, one may refer to Tomar et al. [32], and for more work on a geometrical aspect of a fixed point set, see [12–17, 21, 22, 24, 30, 32, 33, 36].

The motivation for this study stems from the realization that many classical fixed point results, deeply rooted in conventional metric spaces, become inapplicable when extended to the setting of closed and bounded intervals due to structural limitations’ most notably, the absence of additive inverses. The interplay between geometry, topology, and fixed point theory in this setting uncovers new applications, such as the concept of near fixed interval ellipses, potentially relevant in fields like optics, astronomy, and signal processing. This reinforces the mathematical and applied interest in developing and exploring the $b-$interval metric space.

Acknowledging the work of Wu [35], the aim of the present work is to revisit the celebrated Hardy-Rogers contraction [10] and improve quasi-contraction in $b-$interval metric spaces. From this particular $b-$interval metric variant of Hardy-Rogers contraction, we obtain $b-$interval metric variants of some known contractions, for instance, Banach contraction [1], Edelstein contraction [8], Kannan contraction [18], Chatterjea contraction [6], Reich contraction [26], and so on. We illustrate by means of examples that conventional Hardy-Rogers contraction principle [10], and an improved Ćirić contraction principle [5] may not be proved in a $b-$interval metric space, concluding thereby that the celebrated fixed point conclusions may not be proved conventionally in a novel $b-$interval metric space which demonstrates the prominence of a $b-$interval metric space over celebrated distance structures. Moreover, we explore a new direction for the geometric properties of the set of non-unique near-fixed points as an application of $b-$interval metric space by introducing the notion of near fixed interval ellipses and their unique equivalence $\mathcal{E}-$class.

2. Preliminaries

The closed interval [x,y] is the collection of real numbers which is described as $[x,y]=\{z\in\mathbb{R}:x\leq z\leq y\}$. The addition and scalar multiplication on the set $\mathcal{U}$ of closed and bounded intervals in $\mathbb{R}$ is described as:

$$[x,y] \oplus [u,v] = [x+u, y+v], \textit{ and}$$
$$p[x,y] = \begin{cases} [px,py], & p \geq 0 \\ [py,py], & p < 0 \end{cases}, [x,y],[u,v] \in \mathcal{U}.$$

$[0,0] \in \mathcal{U}$ is a zero element of $\mathcal{U}$. For any $[x,y] \in \mathcal{U}$, $[x,y] \ominus [x,y] = [x,y] \oplus [-y,-x] = [x-y, y-x]$. In other words, under the above-defined addition and scalar multiplication operations, $\mathcal{U}$ is not a vector space in the traditional sense since any of its non-degenerate closed intervals may not have an additive inverse.

The null set is therefore defined as follows:

$$\begin{aligned} \mathcal{N} &= \{[x,y] \ominus [x,y] : [x,y] \in \mathcal{U}\} \\ &= \{[-a,a] : a \geq 0\} \\ &= \{a[-1,1] : a \geq 0\}. \end{aligned}$$

More specifically, $[-1,1]$ generates $\mathcal{N}$.

Remark 1 [35]

1. In general, $(a+b)[x,y] \neq a[x,y] + b[x,y]$,
2. If $a,b \geq 0$, $(a+b)[x,y] = a[x,y] + b[x,y]$.
3. If $a,b \leq 0$, $(a+b)[x,y] = a[x,y] + b[x,y]$, $\forall a,b \in \mathbb{R}$.
4. $[x,y] \overset{\mathcal{N}}{=} [u,v]$ iff there exist $n_1, n_2 \in \mathcal{N}$ such that

$$[x,y] + n_1 = [u,v] + n_2.$$

Evidently, $[x,y] = [u,v] \Rightarrow [x,y] + n_1 = [u,v] + n_2, n_1 = n_2 = [0,0] \Rightarrow [x,y] \overset{\mathcal{N}}{=} [u,v]$. The converse may not be fundamentally true, however.

Exploiting the binary relationship $\overset{\mathcal{N}}{=}$, for any $[x,y] \in \mathcal{U}$, we define

$$< [x,y] >= \{[u,v] \in \mathcal{U} : [x,y] \overset{\mathcal{N}}{=} [u,v]\}. \tag{1}$$

The family of all classes $<[x,y]>$ for $[x,y] \in \mathcal{U}$ is symbolized by $<\mathcal{U}>$. According to [35], the binary relation $\overset{\mathcal{N}}{=}$ is an equivalence relation; in other words, the class defined in equation (1) represented the equivalence class. In general, the quotient set of $\mathcal{U}$ is the family $<\mathcal{U}>$ of all the classes (1). It is crucial to note that a quotient set $<\mathcal{U}>$ is likewise not a typical vector space. Notably, $[u,v] \in <[x,y]> \Rightarrow <[x,y]> = <[u,v]>$. As a result, the complete collection of closed and bounded intervals is divided into the family of equivalence classes $\mathcal{U}$ in $\mathbb{R}$.

Definition 2 [35] *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. A metric interval space is the pair $(\mathcal{U}, d)$, on a non-empty set $\mathcal{U}$ iff a map $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ satisfies the subsequent conditions:*

1. $d([x,y],[u,v]) = 0$ iff $[x,y] \overset{\mathcal{N}}{=} [u,v]$;
2. $d([x,y],[u,v]) = d([u,v],[x,y])$;
3. $d([x,y],[u,v]) \leq d([x,y],[r,s]) + d([r,s],[u,v]), [x,y],[r,s],[t,u] \in \mathcal{U}$.

Definition 3 [35] *A metric interval $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ satisfies null equalities, if for $n_1, n_2 \in \mathcal{N}$ and $[x,y],[u,v] \in \mathcal{U}$, the subsequent conditions holds:*

1. $d([x,y] \oplus n_1, [u,v] + n_2) = d([x,y],[u,v])$;
2. $d([x,y] \oplus n_1, [u,v]) = d([x,y],[u,v])$;
3. $d([x,y],[u,v] \oplus n_2) = d([x,y],[u,v])$.

Definition 4 *[35] Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}_{\mathcal{N}}$ be the null set. A point* $[x,y] \in \mathcal{U}$ *is known as a near fixed point of a function* $\mathcal{M} \quad \mathcal{U} \to \mathcal{U}$ *iff* $\mathcal{M}([x,y]) = x,y$.

In [14], the authors introduced the concept of $b-$interval metric space as follows:

Definition 5 *[14] Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. A $b-$interval metric on a non-empty set $\mathcal{U}$ is a map $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ satisfying:*

1. $d_b([x,y],[u,v]) = 0$ iff $[x,y] \stackrel{\mathcal{N}}{=} [u,v]$;
2. $d_b([x,y],[u,v]) = d_b([u,v],[x,y])$;
3. $d_b([x,y],[u,v]) \leq s[d_b([x,y],[r,s]) + d_b([r,s],[u,v])]$, $\mathbf{s} \geq 1$, $\quad [x,y],[r,s],[t,u] \in \mathcal{U}$.

A pair $(\mathcal{U}, d_b)$ is known as a $b-$interval metric space.

A $b-$interval metric space reduces to an interval metric space [35] for $s = 1$.

Following Wu [35], the authors in [14] introduced null equalities in a $b-$interval metric space as follows:

Definition 6 [14] *A $b-$interval metric $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ satisfies $b-$null equalities, if for* $n_1, n_2 \in \mathcal{N}$, *the null set,* $s \geq 1$, *and* $[x,y],[u,v] \in \mathcal{U}$, *the subsequent hypotheses hold:*

1. $d_b([x,y] \oplus n_1, [u,v] + n_2) = d_b([x,y],[u,v])$;
2. $d_b([x,y] \oplus n_1, [u,v]) = d_b([x,y],[u,v])$;
3. $d_b([x,y],[u,v] \oplus n_2) = d_b([x,y],[u,v])$.

Example 7 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ be defined as:*

$$d_b([x,y],[u,v]) = (\log(\frac{x+y}{u+v}))^2. \tag{2}$$

We assert that $(\mathcal{U}, d_b)$ is a $b-$interval metric space and $s = 2$.

1. Let $[x,y],[u,v] \in \mathcal{U}$, $x \leq y$, $u \leq v$. Now,

$$\begin{aligned}
&d_b([x,y],[u,v]) = 0,\\
&\Rightarrow (\log(\frac{x+y}{u+v}))^2 = 0,\\
&\Rightarrow \frac{x+y}{u+v} = 1\\
&\Rightarrow x + y = u + v, \textit{ which is possible iff } \quad y \geq u,\\
&\textit{that is, } x + u - v = 2u - y.
\end{aligned}$$

Since, $x \leq y, u \leq v$, and $y \geq u$, $x + u - v \leq y + v - u$ and $2u - y \leq y + v - u$, we have two identical intervals $[x + u - v, y + v - u]$ and $[2u - y, y + v - u]$. These intervals may be written as

$$[x+u-v, y+v-u] = [x,y] \oplus [u-v, v-u] \text{ and } [2u-y, y+v-u] = [u,v] \oplus [u-y, y-u].$$

Suppose, $n_1 = [u-v, v-u]$ and $n_2 = [u-y, y-u]$, $n_1, n_2 \in \mathcal{N}$.

Now, we have $[x,y] \oplus \mathfrak{n}_1 = [u,v] \oplus \mathfrak{n}_2$. Consequently, $[x,y] \stackrel{\mathcal{N}}{=} [u,v]$.

Conversely, suppose that $[x,y] \stackrel{\mathcal{N}}{=} [u,v]$, then $[x,y] \oplus \mathfrak{n}_1 = [u,v] \oplus \mathfrak{n}_2$, $n_1, n_2 \in \mathcal{N}$,

where, $n_1 = [-(v-u), v-u]$ and $n_2 = [-(y-u), y-u]$.

One may verify that, $d_b([x,y] \oplus \mathfrak{n}_1, [u,v] \oplus \mathfrak{n}_2) = 0$.

2. Since, $d_b([x,y],[u,v]) = (\log(\frac{x+y}{u+v}))^2$

$$= (\log(\frac{u+v}{x+y}))^2$$
$$= d_b([u,v],[x,y]).$$

3. For $[x,y],[u,v],[r,s] \in \mathcal{U}$,

$$\begin{aligned} d_b([x,y],[u,v]) &= (\log(\frac{x+y}{u+v}))^2 \\ &= (\log(\frac{x+y}{r+s} \times \frac{r+s}{u+v}))^2 \\ &= (\log(\frac{x+y}{r+s}) + \log(\frac{r+s}{u+v}))^2 \\ &\le 2[(\log(\frac{x+y}{r+s}))^2 + (\log(\frac{r+s}{u+v}))^2] \\ &= 2[d_b([x,y],[r,s]) + d_b([r,s],[u,v])]. \end{aligned}$$

Hence, d_b is a $b-$interval metric, but d_b is neither an interval metric nor a $b-$metric on $\mathcal{U}$.

Example 8 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. One may verify that d_b is a $b-$interval metric and $w = 2^{p-1}$, but d_b is neither an interval metric nor a $b-$metric on $\mathcal{U}$.*

Remark 9 *It is fascinating to note that for some* $n_1=[-l,l]$ *and* $n_2=[-m,m]$, $l,m \in \mathbb{R}$, *$b-$interval metrics d_b defined in Examples 7 and 8 satisfy $b-$null equalities.*

Example 10 Let $\mathcal{U} = \{[-9,-3],[0,3],[3,9]\}$, $\mathcal{N} = \{[0,0],[-1,1]\}$ and

$$d_b([-9,-3],[3,9]) = d_b([3,9],[-9,-3]) = 5,$$

$$d_b([-9,-3],[0,3]) = d_b([0,3],[-9,-3]) = d_b([0,3],[3,9]) = d_b([3,9],[0,3]) = 1, \text{ and}$$
$$d_b([-9,-3],[-9,-3]) = d_b([0,3],[0,3]) = d_b([3,9],[3,9]) = 0,$$

then one may verify that d_b is a $b-$interval metric and $s = 2.5$ but d_b is neither an interval metric nor a $b-$metric on $\mathcal{U}$.

Since, $d_b([-9,3]+[-1,1],[3,9]) = d_b([-10,4],[3,9])$ is not defined, a $b-$interval metric d_b does not satisfy null equalities.

To study the topology of a $b-$interval metric, $s \ge 1$ and the null set $\mathcal{N}$, the open ball centred at $[x_0,y_0]$ and radius $\varepsilon \in (0,\infty)$ is described as:

$$\mathcal{O}([x_0,y_0],\varepsilon) = \{[x,y] \in \mathcal{U} : d_b([x_0,y_0],[x,y]) < \frac{\varepsilon}{s}\}.$$

The closed ball centred at u and radius $\varepsilon \in (0,\infty)$ is described as:

$$C([x_0,y_0],\varepsilon) = \{[x,y] \in \mathcal{U} : d_b([x_0,y_0],[x,y]) \le \frac{\varepsilon}{s}\}.$$

Lemma 11 [14] *Let $(\mathcal{U},d_b)$ be a $b-$interval metric space, $\mathcal{N}$ be the null set and $s \ge 1$. Then, the collection of all open balls, $\mathcal{O}([x_0,y_0],\varepsilon) = \{[x,y] \in \mathcal{U} : d_b([x_0,y_0],[x,y]) < \frac{\varepsilon}{s}\}$ forms a basis of $(\mathcal{U},d_b)$.*

Theorem 12 [14] *If $(\mathcal{U},d_b)$ is a $b-$interval metric space, $\mathcal{N}$ is the null set, $s \ge 1$ and τ_b is a topology generated by the open ball $\mathcal{O}([x_0,y_0],\varepsilon)$, then $(\mathcal{U},\tau_b)$ is a $\mathcal{T}_0-$space.*

Next, we see the definition of $b-$convergence, $b-$limit, $b-$completeness, continuity, and $b-$Cauchy sequence, $b-$class limit, in the $b-$interval metric space.

Definition 13 [14] *Let* $(\mathcal{U},d_b)$ *be a* $b-$*interval metric space,* $s\geq 1$ *and* $\mathcal{N}$ *be the null set. The sequence* $\{[x_n,y_n]\}_{n=1}^{\infty}$ *in* $\mathcal{U}$ *is said to be* $b-$*convergent iff* $\lim_{n\to\infty}d_b([x_n,y_n],[x,y])=0$, $[x,y]\in\mathcal{U}$. *The element* $[x,y]$ *is known as a* $b-$*limit of the sequence* $\{[x_n,y_n]\}_{n=1}^{\infty}$.

If there exists $[x,y],[u,v]\in\mathcal{U}$ so that $\lim_{n\to\infty}d_b([x_n,y_n],[x,y])=\lim_{n\to\infty}d_b([x_n,y_n],[u,v])=0$, then

$$d_b([x,y],[u,v])\leq s[d_b([x,y],[x_n,y_n])+d_b([x_n,y_n],[u,v])]\to 0,\ as\ n\to\infty, \tag{3}$$

so, by Definition 5, $[x,y]\overset{\mathcal{N}}{=}[u,v]$, that is, $[u,v]\in<[x,y]>$.

Proposition 14 [14] *Let* $\{[x_n,y_n]\}_{n=1}^{\infty}$ *be a sequence in a* $b-$*interval metric space* $(\mathcal{U},d_b)$, $s\geq 1$ *and* $\mathcal{N}$ *be the null set, satisfying* $\lim_{n\to\infty}d_b([x_n,y_n],[x,y])=0$. *Then,*

$$\lim_{n\to\infty}d_b([x_n,y_n],[u,v])=0,\quad for\ any\ \ [u,v]\in<[x,y]>.$$

Definition 15 [14] *Let* $(\mathcal{U},d_b)$ *be a* $b-$*interval metric space,* $\mathcal{N}$ *be the null set and* $s\geq 1$. *If* $\{[x_n,y_n]\}_{n=1}^{\infty}$ *is a sequence in* $\mathcal{U}$ *satisfying* $\lim_{n\to\infty}d_b([x_n,y_n],<[x,y]>)=0,[x,y]\in\mathcal{U}$ *or* $\lim_{n\to\infty}[x_n,y_n]=<[x,y]>$, *then the equivalence class* $<[x,y]>$ *is known as a* $b-$*class limit of the sequence* $\{[x_n,y_n]_{n=1}^{\infty}\}$.

Proposition 16 [14] *The* $b-$*class limit in the* $b-$*interval metric space* $(\mathcal{U},d_b)$, $s\geq 1$ *is unique.*

Definition 17 [14] *Let* $(\mathcal{U},d_b)$ *be a* $b-$*interval metric space,* $\mathcal{N}$ *be the null set and* $s\geq 1$.

1. A sequence $\{[x_n,y_n]\}_{n=1}^{\infty}$ in a $b-$interval metric space $(\mathcal{U},d_b)$ is known as a $b-$Cauchy sequence iff for given $\varepsilon>0$, there exists numbers $n,m,N\in\mathbb{N}$ so that $d_b([x_n,y_n],[x_m,y_m])<\varepsilon$, $n,m>N$.
   Equivalently, $\{[x_n,y_n]\}_{n=1}^{\infty}$ in a topological $b-$interval space $(\mathcal{U},\mathcal{T}_b)$ is known as a $b-$Cauchy sequence iff, for given $\varepsilon>0$, there exists numbers $n,m,N\in\mathbb{N}$ so that $[x_n,y_n],[x_m,y_m]\in\mathcal{O}([x_0,y_0],\varepsilon)$, $n,m>N$.
2. If $\mathcal{V}\subseteq\mathcal{U}$, $(\mathcal{V},d_b)$ is a complete subspace of $(\mathcal{U},d_b)$ iff each $b-$Cauchy sequence in $(\mathcal{V},d_b)$ is $b-$convergent in $(\mathcal{V},d_b)$.

Definition 18 [14] *Let* $(\mathcal{U},d_b)$ *be a* $b-$*interval metric space,* $\mathcal{N}$ *be the null set and* $s\geq 1$. *A self map* $\mathcal{M}\quad\mathcal{U}\to\mathcal{U}$ *is continuous at* $[x_0,y_0]$ *if for every* $\varepsilon>0$ *there exists a number* $\delta(\varepsilon)>0$ *so that*

$$d_b([x,y],[x_0,y_0])<\varepsilon\Rightarrow d_b(\mathcal{M}[x,y],\mathcal{M}[x_0,y_0])<\varepsilon.$$

If $\mathcal{M}$ is continuous at all $[x_0,y_0]\in\mathcal{U}$, then we say that $\mathcal{M}$ is continuous on $\mathcal{U}$.

Example 19 *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. Let* $(\mathcal{U},d_b)$ *be a* $b-$*interval metric space and* $s\geq 1$.

If $\mathcal{M}\quad\mathcal{U}\to\mathcal{U}$ is $\mathcal{M}[x,y]=\begin{cases}4[x,y]+[-1,5], & e\leq 8, f\leq 9\\ 3[x,y]+[10,12], & otherwise\end{cases}$

Then, $\mathcal{M}$ is continuous on $\mathcal{U}$.

If $\mathcal{M}\quad\mathcal{U}\to\mathcal{U}$ is $\mathcal{M}[x,y]=\begin{cases}[x,y], & x,y\in\mho\\ [-y,-x] & otherwise\end{cases}$

Then, $\mathcal{M}$ is continuous only at $[0,0]$.

Again, if $\mathcal{M}\quad\mathcal{U}\to\mathcal{U}$ is $\mathcal{M}[x,y]=\begin{cases}3[x,y]-[2,5], & x,y\leq 0\\ 5[x,y]-[2,5], & 0\leq x<5,\ 0\leq y<7\\ [[5,7]], & otherwise\end{cases}$

Then, $\mathcal{M}$ is continuous on $\mathcal{U}\setminus[5,7]$.

Proposition 20 [14] *Each* $b-$*convergent sequence is a* $b-$*Cauchy sequence in a* $b-$*interval metric space* $(\mathcal{U},d_b)$, $s\geq 1$ *and the null set* $\mathcal{N}$.

The following example demonstrates that the opposite of the above result may not essentially be true, that is, every $b-$Cauchy sequence may not be a $b-$convergent in a $b-$interval metric space.

Example 21 *Let* $\mathcal{U}=\{[x,y]:-2<x,y<2\}$. *Define a* $b-$*interval metric,* $d_b:\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ *as with* $s=2$ *and the null set* $\mathcal{N}$. *Define sequences* $\{[x_n,y_n]\}=\{[-2+\frac{1}{n},2-\frac{1}{n}]\}$ *or* $\{[\frac{-2n^2}{1+n^2},\frac{n}{1+n^2}]\}$ *or* $\{[\frac{1}{n},2+\frac{1}{n}]\}$. *Noticeably, all of these are* $b-$*Cauchy sequences but none of them is* $b-$*convergent sequence.*

3. Main Results

Next, we establish the first main result for a $b-$interval metric variant of Hardy-Roger contraction [10] for determining near fixed points of the function $\mathcal{M}$ and its unique equivalence class.

Theorem 22 *Let* $(\mathcal{U},d_b)$ *be a complete* $b-$*interval metric space satisfying null equalities and* $\mathbf{s}\geq 1$. *Suppose a self map* $\mathcal{M}\quad \mathcal{U}\to\mathcal{U}$ *satisfies*

$$\begin{aligned} d_b(\mathcal{M}[x,y],\mathcal{M}[u,v])\leq\ & \alpha d_b([x,y],[u,v])+\beta d_b([x,y],\mathcal{M}[x,y])+\gamma d_b([u,v],\mathcal{M}[u,v])+\\ & \delta d_b([x,y],\mathcal{M}[u,v])+\eta d_b([u,v],\mathcal{M}[x,y]), \end{aligned} \tag{4}$$

numbers, $\alpha,\beta,\gamma,\delta$, and η are non-negative so that $\alpha+\beta+\gamma+\delta+\eta<\frac{1}{\mathbf{s}}$ *and* $[x,y],[u,v]\in\mathcal{U}$. Then, $\mathcal{M}$ has a near fixed point $[x,y]\in\mathcal{U}$.

Further, $\mathcal{M}$ has a unique equivalence class of near fixed points $<[x,y]>$. Equivalently, if [x,y] and $[\bar{x},\bar{y}]$ are near fixed points of $\mathcal{M}$, then $[x,y]=[\bar{x},\bar{y}]$, that is, $[\bar{x},\bar{y}]\in<[x,y]>$ or $<[x,y]>=<[\bar{x},\bar{y}]>$.

Proof. Given an initial element $[x_0,y_0]\in\mathcal{U}$, the iterative sequence $\{[x_n,y_n]\}_{n=1}^{\infty}$, utilizing the function $\mathcal{M}$, is defined as follows:

$$[x_{n+1},y_{n+1}]=\mathcal{M}([x_n,y_n])=\mathcal{M}^{n+1}([x_0,y_0]). \tag{5}$$

Now, we assert that $\{[x_n,y_n]\}$ is a $b-$convergent sequence, converging to a near fixed point of $\mathcal{M}$ in a $b-$interval metric space. Utilizing (4), we get

$$\begin{aligned} d_b(\ & [x_{n+1},y_{n+1}],[x_n,y_n])=d_b(\mathcal{M}[x_n,y_n],\mathcal{M}[x_{n-1},y_{n-1}])\\ & \leq\alpha d_b([x_n,y_n],[x_{n-1},y_{n-1}])+\beta d_b([x_n,y_n],\mathcal{M}[x_n,y_n])+\gamma d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_{n-1},y_{n-1}])\\ & \quad+\delta d_b([x_n,y_n],\mathcal{M}[x_{n-1},y_{n-1}])+\eta d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_n,y_n])\\ & =\alpha d_b([x_n,y_n],[x_{n-1},y_{n-1}])+\beta d_b([x_n,y_n],[x_{n+1},y_{n+1}])+\gamma d_b([x_{n-1},y_{n-1}],[x_n,y_n])\\ & \quad+\delta d_b([x_n,y_n],[x_n,y_n])+\eta d_b([x_{n-1},y_{n-1}],[x_{n+1},y_{n+1}])\\ & =(\alpha+\gamma)d_b([x_n,y_n],[x_{n-1},y_{n-1}])+\beta d_b([x_{n+1},y_{n+1}],[x_n,y_n])\\ & \quad+\eta\mathbf{s}(d_b([x_{n-1},y_{n-1}],[x_n,y_n])+d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\\ & =(\alpha+\gamma+\eta\mathbf{s})d_b([x_n,y_n],[x_{n-1},y_{n-1}])+(\beta+\eta\mathbf{s})d_b([x_{n+1},y_{n+1}],[x_n,y_n]) \end{aligned} \tag{6}$$

and

$$\begin{aligned} d_b(\ & [x_n,y_n],[x_{n+1},y_{n+1}])=d_b(\mathcal{M}[x_{n-1},y_{n-1}],\mathcal{M}[x_n,y_n])\\ & \leq\alpha d_b([x_{n-1},y_{n-1}],[x_n,y_n])+\beta d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_{n-1},y_{n-1}])+\gamma d_b([x_n,y_n],\mathcal{M}[x_n,y_n])\\ & \quad+\delta d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_n,y_n])+\eta d_b([x_n,y_n],\mathcal{M}[x_{n-1},y_{n-1}])\\ & =\alpha d_b([x_n,y_n],[x_{n-1},y_{n-1}])+\beta d_b([x_{n-1},y_{n-1}],[x_n,y_n])+\gamma d_b([x_n,y_n],[x_{n+1},y_{n+1}])\\ & \quad+\delta d_b([x_{n-1},y_{n-1}],[x_{n+1},y_{n+1}])+\eta d_b([x_n,y_n],[x_n,y_n])\\ & =(\alpha+\beta)d_b([x_n,y_n],[x_{n-1},y_{n-1}])+\gamma d_b([x_{n+1},y_{n+1}],[x_n,y_n])\\ & \quad+\delta\mathbf{s}(d_b([x_{n-1},y_{n-1}],[x_n,y_n])+d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\\ & =(\alpha+\beta+\delta\mathbf{s})d_b([x_n,y_n],[x_{n-1},y_{n-1}])+(\gamma+\delta\mathbf{s})d_b([x_{n+1},y_{n+1}],[x_n,y_n]). \end{aligned} \tag{7}$$

Adding (6) and (7), we get

$$2d_b([x_n,y_n],[x_{n+1},y_{n+1}]) \le (2\alpha+\beta+\gamma+\delta\mathbf{s}+\eta\mathbf{s})d_b([x_n,y_n],[x_{n-1},y_{n-1}]) + (\beta+\gamma+\delta\mathbf{s}+\eta\mathbf{s})$$
$$d_b([x_{n+1},y_{n+1}],[x_n,y_n]),$$

that is, $(2-\beta-\gamma-\delta\mathbf{s}-\eta\mathbf{s})d_b([x_n,y_n],[x_{n+1},y_{n+1}]) \le (2\alpha+\beta+\gamma+\delta\mathbf{s}+\eta\mathbf{s})$
$d_b([x_n,y_n],[x_{n-1},y_{n-1}])$,

which implies, $d_b([x_n,y_n],[x_{n+1},y_{n+1}]) \le \dfrac{(2\alpha+\beta+\gamma+\delta\mathbf{s}+\eta\mathbf{s})}{2-\beta-\gamma-\delta\mathbf{s}-\eta\mathbf{s}} d_b([x_n,y_n],[x_{n-1},y_{n-1}])$,

$$i.e.,\ d_b([x_n,y_n],[x_{n+1},y_{n+1}]) \le \mu d_b([x_n,y_n],[x_{n-1},y_{n-1}]), \tag{8}$$

where $\mu = \dfrac{(2\alpha+\beta+\gamma+\delta\mathbf{s}+\eta\mathbf{s})}{2-\beta-\gamma-\delta\mathbf{s}-\eta\mathbf{s}} < \dfrac{1}{\mathbf{s}}$, since, $\alpha+\beta+\gamma+\delta+\eta < \dfrac{1}{\mathbf{s}}$.

Next, we affirm that $\{[x_n,y_n]\}$ is a $b-$Cauchy sequence. Utilizing the inequality (8), we get

$$\begin{aligned} d_b([x_n,y_n],[x_{n+1},y_{n+1}]) &\le \mu d_b([x_n,y_n],[x_{n-1},y_{n-1}]) \\ &\le \mu^2 d_b([x_{n-1},y_{n-1}],[x_{n-2},y_{n-2}]) \\ &\ \ \vdots \\ &\le \mu^n d_b([x_1,y_1],[x_0,y_0]). \end{aligned}$$

Next, for m>n, we have

$$\begin{aligned} d_b(&[x_n,y_n],[x_m,y_m]) \le s[d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + d_b([x_{n+1},y_{n+1}],[x_m,y_m])] \\ &\le s d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + s^2[d_b([x_{n+1},y_{n+1}],[x_{n+2},y_{n+2}]) + d_b([x_{n+2},y_{n+2}],[x_m,y_m])] \\ &\ \ \vdots \\ &\le s d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + s^2 d_b([x_{n+1},y_{n+1}],[x_{n+2},y_{n+2}]) + s^3 d_b([x_{n+2},y_{n+2}], \\ &\quad [x_{n+3},y_{n+3}]) + \ldots + s^{m-n} d_b([x_{m-1},y_{m-1}],[x_m,y_m]), \\ &\le s\mu^n d_b([x_1,y_1],[x_0,y_0]) + s^2\mu^{n+1} d_b([x_1,y_1],[x_0,y_0]) + \ldots + s^{m-n}\mu^{m-1} d_b([x_1,y_1],[x_0,y_0]) \\ &= s\mu^n[1 + s\mu + (s\mu)^2 + \ldots + (s\mu)^{m-n-1}] d_b([x_1,y_1],[x_0,y_0]) \\ &= s\mu^n\left(\frac{1-(s\mu)^{m-n}}{1-s\mu}\right) d_b([x_1,y_1],[x_0,y_0]) \to 0,\ as\ n\to\infty. \end{aligned}$$

Consequently, the sequence $\{[x_n,y_n]\}_{n=1}^{\infty}$ is a $b-$Cauchy sequence in $\mathcal{U}$. As $(\mathcal{U},d_b)$ is a complete $b-$interval metric space, we have $[x,y]\in\mathcal{U}$ and $d_b([x_n,y_n],[x,y]) \to 0$, that is, $[x_n,y_n] \to [x,y]$, that is, $[x,y] \in <[x,y]>$.

Now, we establish that any $[\bar{x},\bar{y}] \in <[x,y]>$ is a near fixed point of $\mathcal{M}$. Since, $b-$interval metric d_b satisfies null equalities, $[\bar{x},\bar{y}] \oplus n_1 = [x,y] \oplus n_2$, for some $n_1, n_2 \in \mathcal{N}$.

Now,

$$\begin{aligned} &d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]) = d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]\oplus n_1) \\ &\le s[d_b(\mathcal{M}[\bar{x},\bar{y}],[x_n,y_n]) + d_b([x_n,y_n],[\bar{x},\bar{y}]\oplus n_1)] \\ &= s[d_b(\mathcal{M}[\bar{x},\bar{y}],\mathcal{M}[x_{n-1},y_{n-1}]) + d_b([x_n,y_n],[\bar{x},\bar{y}]\oplus n_1)] \\ &\le s[(\alpha d_b([\bar{x},\bar{y}],[x_{n-1},y_{n-1}]) + \beta d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]) + \gamma d_b([x_{n-1},y_{n-1}], \\ &\quad \mathcal{M}[x_{n-1},y_{n-1}]) + \delta d_b([\bar{x},\bar{y}],\mathcal{M}[x_{n-1},y_{n-1}]) + \eta d_b([x_{n-1},y_{n-1}],\mathcal{M}[\bar{x},\bar{y}])) \\ &\quad + d_b([x_n,y_n],[x,y]\oplus n_2)], \end{aligned}$$

which implies

$$\begin{aligned} (1-s\beta)d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]) &\le s[(\alpha d_b([\bar{x},\bar{y}]\oplus n_1,[x_{n-1},y_{n-1}]) + \gamma d_b([x_{n-1},y_{n-1}],[x_n,y_n]) \\ &+ \delta d_b([\bar{x},\bar{y}])\oplus n_1,[x_n,y_n]) + \eta\mathbf{s}(d_b([x_{n-1},y_{n-1}],[\bar{x},\bar{y}]) \\ &+ d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]))) + d_b([x_n,y_n],[x,y])]. \end{aligned}$$

Therefore, we get

$$\begin{aligned}(1-s\beta-s^2\eta)d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]) &\le s[\alpha d_b([x,y]\oplus n_2,[x_{n-1},y_{n-1}])+\gamma d_b([x_{n-1},y_{n-1}],[x_n,y_n])\\ &\quad+\delta d_b([x,y])\oplus n_2,[x_n,y_n])+\eta s d_b([x_{n-1},y_{n-1}],[\bar{x},\bar{y}]\oplus n_1)\\ &\quad+d_b([x_n,y_n],[x,y])]\\ &= s[\alpha d_b([x,y],[x_{n-1},y_{n-1}])+\gamma d_b([x_{n-1},y_{n-1}],[x_n,y_n])+\\ &\quad\delta d_b([x,y],[x_n,y_n])+\eta s d_b([x_{n-1},y_{n-1}],[\bar{x},\bar{y}]\oplus n_1)\\ &\quad+d_b([x_n,y_n],[x,y])]\quad as\ \ n\to\infty,\end{aligned}$$

that is, $\mathcal{M}[\bar{x},\bar{y}]\overset{\mathcal{N}}{=}[\bar{x},\bar{y}]$, for $[\bar{x},\bar{y}]\in<[x,y]>$.

Suppose [x,y] and [u,v] are two different near fixed points of $\mathcal{M}$ so that $[x,y]\in<[x,y]>$ and $[u,v]\notin<[x,y]>$, however [u,v] belongs to some different equivalence classes. So, $\mathcal{M}[x,y]\overset{\mathcal{N}}{=}[x,y]$ and $\mathcal{M}[u,v]\overset{\mathcal{N}}{=}[u,v]$. Then, $\mathcal{M}[x,y]\oplus n_1=[x,y]\oplus n_2$ and $\mathcal{M}[u,v]\oplus n_3=[u,v]\oplus n_4$, for some $n_1,n_2,n_3,n_4\in\mathcal{N}$. Now,

$$\begin{aligned}d_b([x,y],[u,v]) &= d_b([x,y]\oplus n_2,[u,v]\oplus n_4)\\ &= d_b(\mathcal{M}[x,y]\oplus n_1,\mathcal{M}[u,v]\oplus n_3)\\ &= d_b(\mathcal{M}[x,y],\mathcal{M}[u,v])\\ &\le \alpha d_b([x,y],[u,v])+\beta d_b([x,y],\mathcal{M}[x,y])+\gamma d_b([u,v],\mathcal{M}[u,v])+\delta d_b([x,y],\mathcal{M}[u,v])\\ &\quad+\eta d_b([u,v],\mathcal{M}[x,y])\\ &= \alpha d_b([x,y],[u,v])+\beta d_b([x,y],[x,y])+\gamma d_b([u,v],[u,v])+\delta d_b([x,y],[u,v])+\\ &\quad\eta d_b([u,v],[x,y])\\ &= \alpha d_b([x,y],[u,v])+\delta d_b([x,y],[u,v])+\eta d_b([u,v],[x,y])\\ &= \alpha d_b([x,y],[u,v])+\delta d_b([x,y],[u,v])+\eta d_b([u,v],[x,y])\\ &= (\alpha+\delta+\eta)d_b([u,v],[x,y])\\ (1-\alpha-\delta-&\eta)d_b([u,v],[x,y])\le 0,\end{aligned}$$

a contradiction. Hence, $[u,v]\in<[x,y]>$, concluding thereby that $<[x,y]>$ is a unique equivalence class of near fixed points of a self map $\mathcal{M}$.

To demonstrate the effectiveness and robustness of our new $b-$interval metric in establishing an environment for the survival of close fixed points and its distinct equivalence class for both continuous and discontinuous maps, we then provide an illustrative example.

Example 23 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Define a $b-$interval metric, $d_b:\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ on $\mathcal{U}$ as $d_b([x,y],[u,v])=(x+y-u-v)^2$. Then, $(\mathcal{U},d_b)$ is a complete $b-$interval metric space and $s=2$. Now, if a map $\mathcal{M}\quad\mathcal{U}\to\mathcal{U}$ is $\mathcal{M}[x,y]=[\frac{5}{13}x,\frac{5}{13}y]$.*

Observe that, for $[x,y],[u,v]\in\mathcal{U}$,

$$\begin{aligned}d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) &= d_b([\tfrac{5}{13}x,\tfrac{5}{13}y],[\tfrac{5}{13}u,\tfrac{5}{13}v])\\ &= \frac{25}{169}(x+y-u-v)^2\\ &= \frac{25}{169}d_b([x,y],[u,v])\\ &\le \frac{25}{169}d_b([x,y],[u,v])+\frac{1}{169}d_b([x,y],\mathcal{M}[x,y])+\frac{1}{169}d_b([u,v],\mathcal{M}[u,v])+\\ &\quad\frac{1}{169}d_b([x,y],\mathcal{M}[u,v])+\frac{1}{169}d_b([u,v],\mathcal{M}[x,y]),\end{aligned}$$

that is, $\mathcal{M}$ satisfies inequality (4) for $\alpha = \frac{25}{169}$ and $\beta = \gamma = \delta = \eta = \frac{1}{169}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $<[0,0]>$, and $[-1,1] \overset{\mathcal{N}}{=} [0,0]$. Noticeably, $\mathcal{M}$ has infinitely many near fixed points.

Example 24 *Let $\mathcal{U}$ be the set of closed and bounded intervals defined on $\{[x,y]:[x,y] \subseteq [0,1]\}$ and $\mathcal{N}$ be the null set. Define a $b-$interval metric, $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ on $\mathcal{U}$ as $d_b([x,y],[u,v]) = | x + y - u - v |^3$. Then, $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space and $s = 4$. Now, define a map $\mathcal{M} \quad \mathcal{U} \to \mathcal{U}$ as*

$$\mathcal{M}[x,y] = \begin{cases} [0,0], & [x,y] \subseteq \left[0, \frac{1}{2}\right] \\ \left[\frac{1}{50}, \frac{1}{25}\right] & \text{otherwise} \end{cases}.$$

It is easy to verify that $\mathcal{M}$ satisfies inequality (4), when $[x,y]$, $[u,v] \subseteq [0,\frac{1}{2}]$ or $[x,y]$, $[u,v] \not\subseteq [0,\frac{1}{2}]$.

Now, when $[x,y] \subseteq [0,\frac{1}{2}]$ and $[u,v] \not\subseteq [0,\frac{1}{2}]$. In particular, let $[x,y] = [\frac{1}{4},\frac{1}{3}]$ and $[u,v] = [\frac{1}{6},\frac{10}{9}]$, then

$$\begin{aligned} d_b(\mathcal{M}[x,y], \mathcal{M}[u,v]) &= d_b([0,0],[\frac{1}{50},\frac{1}{25}]) \\ &= | 0 + 0 - \frac{1}{50} - \frac{1}{25} |^3 \\ &\leq \frac{1}{60} d_b([x,y],[u,v]) + \frac{1}{80} d_b([x,y],\mathcal{M}[x,y]) + \frac{1}{80} d_b([u,v],\mathcal{M}[u,v]) \\ &\quad + \frac{1}{60} d_b([x,y],\mathcal{M}[u,v]) + \frac{1}{40} d_b([u,v],\mathcal{M}[x,y]), \end{aligned} \tag{9}$$

that is, $\mathcal{M}$ satisfies inequality (4) for $\alpha = \frac{1}{60}, \beta = \frac{1}{80}, \gamma = \frac{1}{80}, \delta = \frac{1}{60}, \eta = \frac{1}{40}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $<[0,0]>$ and $[-1,1] \overset{\mathcal{N}}{=} [0,0]$. Noticeably, $\mathcal{M}$ has infinitely many near fixed points.

Next, we exploit $b-$interval metric variants of the Banach contraction [1], Chatterjea contraction [6], Edelstein contraction [8], Kannan contraction [18], and Reich contraction [26] for determining near fixed points of the function $\mathcal{M}$ and its equivalence class.

Theorem 25 *Theorem 22 still holds true, if inequality (3) is substituted by a $b-$interval metric variant of Banach contraction:*

$$d_b(\mathcal{M}[x,y], \mathcal{M}[u,v]) \leq \alpha d_b([x,y],[u,v]) \quad 0 \leq \alpha < \frac{1}{\mathbf{s}}. \tag{10}$$

Proof. The proof is almost analogous to the proof of Theorem 22.

Remark 26 *If we substitute $\beta = \gamma = \delta = \eta = 0$ in inequality (3), Theorem 25 is a particular type of Theorem 22.*

Theorem 27 *Theorem 22 still holds true, if inequality (3) is substituted by a $b-$interval metric variant of Chatterjea contraction:*

$$d_b(\mathcal{M}[x,y], \mathcal{M}[u,v]) \leq \eta(d_b([x,y],\mathcal{M}[u,v]) + d_b([u,v],\mathcal{M}[x,y])), \quad 0 \leq \eta < \frac{1}{2\mathbf{s}}. \tag{11}$$

Proof. The proof is almost analogous to the proof of Theorem 22.

Remark 28 *If we substitute $\alpha=\beta=\gamma=0$ and $\eta=\delta$ in inequality* (3)*, Theorem 27 is a particular type of Theorem 22.*

Theorem 29 *Theorem 22 still holds true, if inequality (3) is substituted by a $b-$interval metric variant of Edelstein contraction:*

$$d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) < d_b([x,y],[u,v]). \tag{12}$$

Proof. The proof is almost analogous to the proof of Theorem 22.

Remark 30 *If we substitute $\alpha=1$ and $\beta=\gamma=\delta=\eta=0$ in inequality* (3)*, Theorem 29 is a particular type of Theorem 22.*

Theorem 31 *Theorem 22 still holds true, if inequality (3) is substituted by a $b-$interval metric variant of Kannan contraction:*

$$d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) \le \beta(d_b([x,y],\mathcal{M}[x,y]) + d_b([u,v],\mathcal{M}[u,v])), \quad 0\le\beta<\frac{1}{2s}. \tag{13}$$

Proof. The proof is almost analogous to the proof of Theorem 22.

Remark 32 *If we substitute $\alpha=\delta=\eta=0$ and $\beta=\gamma$ in inequality* (3)*, Theorem 31 is a particular type of Theorem 22.*

Theorem 33 *Theorem 22 still holds true, if inequality (3) is substituted by a $b-$interval metric variant of Reich contraction:*

$$d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) \le \alpha d_b([x,y],[u,v]) + \beta d_b([x,y],\mathcal{M}[x,y]) + \gamma d_b([u,v],\mathcal{M}[u,v]), \tag{14}$$

$\alpha+\beta+\gamma<\frac{1}{s}$, and numbers α,β,γ are non-negative.

Proof. The proof is almost analogous to the proof of Theorem 22.

Remark 34 *If we substitute $\delta=\eta=0$ in inequality* (3)*, Theorem 33 is a particular type of Theorem 22.*

Next, we present an improved $b-$interval metric variant of Theorem 1 of C′iric′ [5] which is an extension of Banach [1], Bianchini [2], C′iric′ [4], Edelstein [7], Kannan [18], Rakotch [25], Reich [26], Sehgal [28], Zamfirescu [37] and so on.

Theorem 35 *Theorem 22 still holds true, if (3) is substituted by the following $b-$interval metric variant of quasi contraction:*

$$\begin{aligned} d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) \le \mu\max\{d_b([x,y],[u,v]), \quad & d_b([x,y],\mathcal{M}[x,y]), d_b([u,v],\mathcal{M}[u,v]), \\ & d_b([x,y],\mathcal{M}[u,v]), d_b([u,v],\mathcal{M}[x,y])\}, \end{aligned} \tag{15}$$

$$\mu\in[0,\frac{1}{s}) \text{ and } [x,y],[u,v]\in\mathcal{U}.$$

Proof. Let the sequence $\{[x_n,y_n]\}$ be defined as in Theorem 22.

Now,

$$\begin{aligned} d_b([x_{n+1},y_{n+1}],[x_n,y_n]) &= d_b(\mathcal{M}[x_n,y_n],\mathcal{M}[x_{n-1},y_{n-1}]) \\ &\le \mu \max\{d_b([x_n,y_n],[x_{n-1},y_{n-1}]), d_b([x_n,y_n],\mathcal{M}[x_n,y_n]), \\ &\quad d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_{n-1},y_{n-1}]), d_b([x_n,y_n],\mathcal{M}[x_{n-1},y_{n-1}]), \\ &\quad d_b([x_{n-1},y_{n-1}],\mathcal{M}[x_n,y_n])\} \\ &= \mu \max\{d_b([x_n,y_n],[x_{n-1},y_{n-1}]), d_b([x_n,y_n],[x_{n+1},y_{n+1}]), \\ &\quad d_b([x_{n-1},y_{n-1}],[x_n,y_n]), d_b([x_n,y_n],[x_n,y_n]), \\ &\quad d_b([x_{n-1},y_{n-1}],[x_{n+1},y_{n+1}])\} \\ &= \mu \max\{d_b([x_n,y_n],[x_{n+1},y_{n+1}]), d_b([x_{n-1},y_{n-1}],[x_n,y_n]), \\ &\quad d_b([x_{n-1},y_{n-1}],[x_{n+1},y_{n+1}])\} \\ &= \mu \max\{d_b([x_n,y_n],[x_{n+1},y_{n+1}]), d_b([x_{n-1},y_{n-1}],[x_n,y_n]), \\ &\quad s(d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\}. \end{aligned}$$

We discuss three cases:

If $\max\{d_b([x_n,y_n],[x_{n+1},y_{n+1}]), d_b([x_{n-1},y_{n-1}],[x_n,y_n]), s(d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\} = d_b([x_n,y_n],[x_{n+1},y_{n+1}])$, then

$d_b([x_{n+1},y_{n+1}],[x_n,y_n]) \le \eta d_b([x_n,y_n],[x_{n+1},y_{n+1}])$, a contradiction.

If $\max\{d_b([x_n,y_n],[x_{n+1},y_{n+1}]), d_b([x_{n-1},y_{n-1}],[x_n,y_n]), s(d_b([x_{n-1},y_{n-1}],[x_n,y_n])$

$$\begin{aligned} &+d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\} \\ &= s[d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([x_n,y_n],[x_{n+1},y_{n+1}])], \end{aligned}$$

then

$$\begin{aligned} d_b([x_{n+1},y_{n+1}],[x_n,y_n]) &\le \mu s[d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([e_n,y_n],[x_{n+1},y_{n+1}])] \\ &< d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([x_n,y_n],[x_{n+1},y_{n+1}]) \\ 0 &< d_b([x_{n-1},y_{n-1}],[x_n,y_n]), \textit{ which is trivial.} \end{aligned}$$

If $\max\{d_b([x_n,y_n],[x_{n+1},y_{n+1}]), d_b([x_{n-1},y_{n-1}],[x_n,y_n]), s(d_b([x_{n-1},y_{n-1}],[x_n,y_n]) + d_b([x_n,y_n],[x_{n+1},y_{n+1}]))\} = d_b([x_{n-1},y_{n-1}],[x_n,y_n])$, then

$d_b([x_{n+1},y_{n+1}],[x_n,y_n]) \le \mu d_b([x_{n-1},y_{n-1}],[x_n,y_n])$, which is same as (8).

Thus, the sequence $\{[x_n,y_n]\}_{n=1}^{\infty}$ verifies all the hypotheses of Theorem 22. So, following similar steps as in Theorem 22, we come to conclusion that $\mathcal{U}$ has a near fixed point and a unique equivalence class of near fixed points $<[x,y]>$.

In order to emphasize the significance of Theorem 1 of C′iric′ [5] not being true in a $b-$interval metric space and Theorem 35 being valid for both continuous and discontinuous mappings, the following examples are provided.

Example 36 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Define a $b-$interval metric, $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ on $\mathcal{U}$ as $d_b([x,y],[u,v]) = |x + y - u - v|^3$. Then, $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space and $s = 4$. Now, define a map $\mathcal{M}$ $\mathcal{U} \to \mathcal{U}$ as $\mathcal{M}[x,y] = [\frac{3}{10}x, \frac{3}{10}y]$. Observe that, for $[x,y],[u,v] \in \mathcal{U}$,*

$$d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) = d_b([\frac{3}{10}x, \frac{3}{10}y],[\frac{3}{10}u, \frac{3}{10}v])$$

$$= (\frac{3}{10}x + \frac{3}{10}y - \frac{3}{10}u - \frac{3}{10}v)^3$$
$$= \frac{9}{100}|x+y-u-v|^3$$
$$\le \frac{9}{100}\max\{d_b([x,y],[u,v]), d_b(\mathcal{M}[x,y],[u,v]), d_b(\mathcal{M}[x,y],[u,v]),$$
$$d_b([x,y],\mathcal{M}[u,v])\},$$

that is, $\mathcal{M}$ satisfies inequality (15) for $\eta = \frac{9}{100}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $<[-1,1]>$, a near fixed point $[0,0]$ and $[0,0] \stackrel{\mathcal{N}}{=} [-1,1]$. Noticeably, $\mathcal{M}$ has infinitely many near fixed points.

Example 37 *Let* $\mathcal{U}$ *be the set of closed and bounded intervals defined on* $\{[x,y]:[x,y] \subseteq [-1,1]\}$ *and* $\mathcal{N}$ *be the null set. Define a* $b-$*interval metric,* $d_b : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ *on* $\mathcal{U}$ *as*

$$d_b([x,y],[u,v]) = (x+y-u-v)^2.$$

Then, $(\mathcal{U}, d_b)$ is a complete $b-$interval metric space and $s = 2$. Now, if a map $\mathcal{M} \quad \mathcal{U} \to \mathcal{U}$ is

$$\mathcal{M}[x,y] = \begin{cases} [\frac{1}{22}, \frac{1}{19}], & [x,y] \subseteq [-1, -\frac{1}{2}] \\ [0,0], & otherwise \end{cases}.$$

Noticeably, $\mathcal{M}$ verifies inequality (15), when $[x,y]$, $[u,v] \subseteq [-1, -\frac{1}{2}]$ or $[x,y]$, $[u,v] \not\subseteq [-1, -\frac{1}{2}]$.

Now, when $[x,y] \subseteq [0, \frac{1}{2}]$ and $[u,v] \not\subseteq [0, \frac{1}{2}]$. In particular, let $[x,y] = [\frac{1}{21}, \frac{1}{23}]$ and $[u,v] = [\frac{1}{3}, \frac{1}{2}]$, then

$$\begin{aligned} d_b(\mathcal{M}[x,y], \mathcal{M}[u,v]) &= d_b([\frac{1}{22}, \frac{1}{19}], [0,0]) \\ &= (\frac{1}{22} + \frac{1}{19} - 0 - 0)^2 \\ &\le \frac{1}{100}\max\{d_b([x,y],[u,v]), d_b([x,y], \mathcal{M}[x,y]), d_b([u,v], \mathcal{M}[u,v]), \\ &\quad d_b([x,y], \mathcal{M}[u,v]), d_b([u,v], \mathcal{M}[x,y])\}, \end{aligned}$$

that is, $\mathcal{M}$ satisfies inequality (15) for $\mu = \frac{1}{100}$. Hence, $\mathcal{M}$ has a unique equivalence class of near fixed points $<[0,0]>$, and $[-4,4] \stackrel{\mathcal{N}}{=} [0,0]$. Noticeably, $\mathcal{M}$ has infinitely many near fixed points.

Example 38 *Let* $X = [0,1] \subset R$*, and define a mapping* $T : X \to X$ *by*

$$T(x) = \begin{cases} \frac{x}{2}, \text{ if } & x \in [0,1), \\ 1, \text{ if } & x = 1. \end{cases}$$

Now consider the usual metric $d(x,y) = |x-y|$. In this setting, the mapping T is not continuous at $x = 1$. T does not satisfy the Banach contraction condition: There exists $0 < k < 1$ such that $d(Tx, Ty) \le kd(x,y)$ for all $x, y \in X$. Consequently, classical fixed point results like Banach's contraction principle, Kannan's, or Hardy–Rogers' principles are not applicable, as these require continuity or strict contractiveness. But in the $b-$interval metric space: Define a $b-$interval metric $D: X \times X \to I \subset R$ by $D(\mathrm{x},\mathrm{y}) = [min\{\mathrm{x},\mathrm{y}\}, max\{\mathrm{x},\mathrm{y}\}]$, where I denotes the set of all closed and bounded intervals in R. With an appropriate null set $N \subset I$ (e.g., intervals of the form $[a,a]$), we can verify: D satisfies the conditions of a $b-$interval metric.

The map T satisfies a Hardy–Rogers type contraction in the sense of the newly defined $b-$interval metric (e.g., using interval-based comparisons involving multiple arguments like $D(Tx,Ty) \le \alpha D(\mathrm{x},\mathrm{y}) + \beta D(\mathrm{x},T\mathrm{x}) + \ldots$). Hence a near fixed point exists, the equivalence class of near fixed points is non-empty, Continuity of T is not required, classical metric space results fail, but the $b-$interval framework succeed.

Remark 39

1. It is fascinating to note that Examples 23-37 do not hold true for near fixed point theorem 1 of Wu [35] and consequently, Theorems 22-35 are genuine extensions and generalizations of Theorem 1 of Wu [35] to $b-$interval metric space.
2. It is interesting to see that a closed and bounded interval $[\mathrm{x},\mathrm{y}] \in \mathcal{U}$ is a fixed point of a self map $\mathcal{M}$ in a $b-$interval metric space iff $\mathcal{M}[\mathrm{x},\mathrm{y}]=[\mathrm{x},\mathrm{y}]$. The concept of this fixed point in a $b-$interval metric space is entirely dissimilar to the fixed point for set-valued maps. So we may conclude that we cannot study the fixed points of celebrated contractions in a conventional way. Instead, we study the near fixed point. Noticeably, in Examples 23-37, a self map $\mathcal{M}$ has infinitely many near fixed points, which are in its unique equivalence class. On the other hand $[0,0]$ is a unique fixed point of $\mathcal{M}$.
3. Near fixed point conclusions established (Theorems 22-35) in a $b-$interval metric space may be utilized to investigate the solutions of the real-world mathematical problems involving the interval-valued maps and maybe a topic of research in the time to come.

In the next result, we establish that a $b-$interval metric variant of an improved quasi-contraction characterizes the completeness of a $b-$interval metric space.

Theorem 40 *If each self-map $\mathcal{M}$ of a $b-$interval metric space $(\mathcal{U},d_b)$ satisfying an inequality (15) of Theorem 35 has a near fixed point, then $(\mathcal{U},d_b)$ is a complete $b-$interval metric space.*

Proof. Let every self-map $\mathcal{M}$ of $\mathcal{U}$ verifying inequality (15) of Theorem 35 has a near fixed point. We affirm that $(\mathcal{U},d_b)$ is complete.

If a $b-$interval metric space $(\mathcal{U},d_b)$ is not complete, we have a $b-$Cauchy sequence in $\mathcal{U}$□ $\mathcal{V} = \{[\mathrm{x}_1,\mathrm{y}_1],[\mathrm{x}_2,\mathrm{y}_2],\ldots,[\mathrm{x}_\mathrm{n},\mathrm{y}_\mathrm{n}],\ldots\}$, say, which consist of distinct points of $\mathcal{U}$ but is not $b-$convergent in $(\mathcal{U},d_b)$.

Let $[\mathrm{z}_1,\mathrm{z}_2] \in \mathcal{U}$ such that $[\mathrm{z}_1,\mathrm{z}_2]$ is not a limit point of the sequence $\mathcal{V}$, that is, $d_b([\mathrm{x},\mathrm{y}],\mathcal{V} \setminus [\mathrm{x},\mathrm{y}]) > 0$ and we have a least positive integer $N([\mathrm{z}_1,\mathrm{z}_2])$ so that $[\mathrm{z}_1,\mathrm{z}_2] \ne [\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}]$. Also, for every $m \ge N([\mathrm{z}_1,\mathrm{z}_2])$, we have

$$d_b([\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}],[\mathrm{x}_\mathrm{m},\mathrm{y}_\mathrm{m}]) << \mu d_b([\mathrm{z}_1,\mathrm{z}_2],[\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}]). \tag{16}$$

Let us define a map $\mathcal{M}\quad \mathcal{U} \to \mathcal{U}$ by $\mathcal{M}[\mathrm{z}_1,\mathrm{z}_2] = [\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}]$. Then, $\mathcal{M}[\mathrm{z}_1,\mathrm{z}_2] \overset{\mathcal{N}}{\ne} [\mathrm{z}_1,\mathrm{z}_2]$ for every $[\mathrm{z}_1,\mathrm{z}_2]$. Using inequality (16), for any $[\mathrm{z}_1,\mathrm{z}_2],[\mathrm{u}_1,\mathrm{u}_2] \in \mathcal{M}$, we get

If $N([\mathrm{z}_1,\mathrm{z}_2]) \le N([\mathrm{u}_1,\mathrm{u}_2])$

$$\begin{aligned} d_b(\mathcal{M}[\mathrm{z}_1,\mathrm{z}_2],\mathcal{M}[\mathrm{u}_1,\mathrm{u}_2]) &= d_b([\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}],[\mathrm{x}_{N([\mathrm{u}_1,\mathrm{u}_2])},\mathrm{y}_{N([\mathrm{u}_1,\mathrm{u}_2])}]) \\ &<< \mu d_b([\mathrm{z}_1,\mathrm{z}_2],[\mathrm{x}_{N([\mathrm{z}_1,\mathrm{z}_2])},\mathrm{y}_{N([\mathrm{z}_1,\mathrm{z}_2])}]), \end{aligned}$$

$$\text{that is,}\quad d_b(\mathcal{M}[z_1,z_2],\mathcal{M}[u_1,u_2]) << \mu d_b([z_1,z_2],\mathcal{M}[z_1,z_2]). \tag{17}$$

If $N([z_1,z_2]) \geq N([u_1,u_2])$, then

$$d_b(\mathcal{M}[z_1,z_2],\mathcal{M}[u_1,u_2]) << \mu d_b([u_1,u_2],\mathcal{M}[u_1,u_2]). \tag{18}$$

Inequalities (16) and (18), implies that

$$\begin{aligned} d_b(\mathcal{M}[z_1,z_2],\mathcal{M}[u_1,u_2]) \quad & << \mu \max\{d_b([z_1,z_2],[u_1,u_2]), d_b([z_1,z_2],\mathcal{M}[z_1,z_2]), d_b(\mathcal{M}[u_1,u_2], \\ & \mathcal{M}[u_1,u_2]), d_b([z_1,z_2],\mathcal{M}[u_1,u_2]), d_b(\mathcal{M}[u_1,u_2],\mathcal{M}[z_1,z_2])\}, \end{aligned} \tag{19}$$

that is, $\mathcal{M}$ satisfies condition (15) of Theorem 35. But $\mathcal{M}$ does not have a near fixed point and its range is a subset of $\mathcal{V}$. Consequently, there exists no sequence $[x_n,y_n]$ in $\mathcal{U}$ for which $\mathcal{M}[x_n,y_n]$ converges. Thus, a self-map $\mathcal{M}$ of $(\mathcal{U},d_b)$ verifies all the postulates of Theorem 35, but does not have a near fixed point, a contradiction. Hence, $(\mathcal{U},d_b)$ is complete.

4. Application

Given that the $b-$interval metric space $(\mathcal{U},d_b)$, under discussion is not a metric space, we are unable to examine the fixed ellipse proposed by Joshi et al. [16] on a $b-$interval metric space $(\mathcal{U},d_b)$ in a conventional manner. The geometry of the set of non-unique near-fixed points on a map will therefore be investigated in relation to a so-called near-fixed interval ellipse, and its equivalence class will be defined as the equivalence $\mathcal{E}-$class of interval ellipses.

In a $b-$interval metric space, we define an interval ellipse as follows:

Definition 41 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let $(\mathcal{U},d_b)$ be a $b-$interval metric space and $s \geq 1$. An interval ellipse $\mathcal{E}([x_1,y_1],[x_1,y_1],a)$ having foci at $[x_1,y_1],[x_2,y_2]$ and length of major axis $\mathbf{a}$ is defined as:*

$$\begin{aligned} \mathcal{E}([x_1,y_1],[x_2,y_2],a) = \{[x,y] \in \mathcal{U} : \quad & d_b([x,y],[x_1,y_1]) + d_b([x,y],[x_2,y_2]) = a, \\ & [x_1,y_1],[x_2,y_2] \in \mathcal{U}, a \in [0,\infty)\}. \end{aligned} \tag{20}$$

If $c_1=[x_1,y_1]$ and $c_2=[x_2,y_2]$, the midpoint of a line c_1c_2 is known as a centre of an interval ellipse. In this case, the major axis of an interval ellipse is the segment of length $\mathbf{a}$ on line c_1c_2, while the minor axis is the line perpendicular to it through the centre. Additionally, the length of an interval ellipse's semi-major axis is $\frac{\mathbf{a}}{2}$. The linear eccentricity is $2f = d(c_1,c_2)$. An ellipse's (ellipsoid's) numerical eccentricity is $\varepsilon = \frac{sinf}{sin\mathbf{a}}$.

In the interval metric space, the interval ellipse for $s=1$ is (20). Additionally, an ellipse in an interval is not always the same as an ellipse in a Euclidean space.

Example 42 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a $b-$interval metric $d_b : \mathcal{U} \to \mathcal{U}$ be defined as $d_b([x,y],[u,v]) = |x+y-u-v|^3$ with $s=4$. Then, an interval ellipse having foci at $c_1=[1,9]$, $c_2=[2,5] \in \mathcal{U}$ and length of major axis $a=12$ is*

$$\begin{aligned} \mathcal{E}([1,9],[2,5],12) \quad & = \{[x,y] \in \mathcal{U} : d_b([x,y],[1,9]) + d_b([x,y],[2,5]) = 12\} \\ & = \{x,y] \in \mathcal{U} : | x+y-1-9 |^3 + | x+y-2-5 |^3 = 12\} \\ & = \{x,y] \in \mathcal{U} : | x+y-10 |^3 + | x+y-7 |^3 = 12\}. \end{aligned}$$

For any $[u,v] \in <[1,9]>$, $[t,u] \in <[2,5]>$, $\mathcal{E}([1,9],[2,5],12) = \mathcal{E}([u,v],[t,u],12)$. So, we define a $\mathcal{E}-$class of interval ellipses having foci at $[x_0,y_0]$, $[u_0,v_0]$ and length of the major axis a, using a binary relation $\overset{\mathcal{R}_x}{\approx}$ as:

$$<\mathcal{E}([x_0,y_0],[u_0,v_0],a)>= \{\mathcal{E}([x,y],[u,v],a):\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_x}{\approx}\mathcal{E}([x_0,y_0],[u_0,v_0],a),$$
$$if\ \mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x_0,y_0],[u_0,v_0],a),\quad [x,y]\overset{\mathcal{N}}{=}[x_0,y_0] \tag{21}$$
$$and\ \ [u,v]\overset{\mathcal{N}}{=}[u_0,v_0],\ [x,y],[u,v],[x_0,y_0],[u_0,v_0]\in\mathcal{U}\}.$$

Let $\breve{E}$ symbolizes the collection of all $\mathcal{E}-$classes of interval ellipses defined on the elements of $\mathcal{U}$.

Proposition 43 *The binary relation* $\overset{\mathcal{R}_x}{\approx}$ *is an equivalence relation.*

Proof.

1. For $\mathcal{E}([x,y],[u,v],a)\in\breve{E}$, $\mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x,y],[u,v],a)$

for $n_1=n_2=[0,0]$, $[x,y]\overset{\mathcal{N}}{\approx}[x,y]$, $[u,v]\overset{\mathcal{N}}{\approx}[u,v]$, so $\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_x}{\approx}\mathcal{E}([x,y],[u,v],r)$, which shows the reflexivity.

2. Let $\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_x}{\approx}\mathcal{E}([x_1,y_1],[u_1,v_1],a)$,
that is, $\mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x_1,y_1],[u_1,v_1],a)$ and $[x,y]\oplus n_1=[x_1,y_1]+n_2$, $[u,v]\oplus n_3=[u_1,v_1]+n_4$,
or $\mathcal{E}([x_1,y_1],[u_1,v_1],a)=\mathcal{E}([x,y],[u,v],a)$ and $[x_1,y_1]\oplus n_2=[x,y]+n_1$, $[u_1,v_1]\oplus n_4=[u,v]+n_3$,
$\Rightarrow\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_x}{\approx}\mathcal{E}([x_1,y_1],[u_1,v_1],a)$, which shows the symmetry.
3. Let $\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_e}{\approx}\mathcal{E}([x_1,y_1],[u_1,v_1],a)$ and $\mathcal{E}([x_1,y_1],[u_1,v_1],a)\overset{\mathcal{R}_e}{\approx}\mathcal{E}([x_2,y_2],[u_2,v_2],a)$. We assert that $\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_e}{\approx}\mathcal{E}([x_2,y_2],[u_2,v_2],a)$.
Since, $\mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x_1,y_1],[u_1,v_1],a)$, $[x,y]+n_1=[x_1,y_1]+n_2$, $[u,v]+n_3$
$=[u_1,v_1]+n_4$ and since, $\mathcal{E}([x_1,y_1],[u_1,v_1],a)=\mathcal{E}([x_2,y_2],[u_2,v_2],a)$, $[x_1,y_1]+n_5$
$=[x_2,y_2]+n_6$, $[u_1,v_1]+n_7=[u_2,v_2]+n_8$, for some $n_i\in\mathcal{N}$, $i=1,2,\ldots,8$.
Hence, $\mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x_2,y_2],[u_2,v_2],a)$,
$[x,y]+n_1+n_5=[x_1,y_1]+n_2+n_5=[x_2,y_2]+n_2+n_6$ and
$[u,v]+n_3+n_7=[u_1,v_1]+n_4+n_7=[u_2,v_2]+n_4+n_8$,
$$\Rightarrow\mathcal{E}([x,y],[u,v],a)\overset{\mathcal{R}_e}{\approx}\mathcal{E}([x_2,y_2],[u_2,v_2],a),\text{ which shows transitivity.}$$

Accordingly, the equivalence $\mathcal{E}-$class of interval ellipses is the $\mathcal{E}-$class of interval ellipses established above in (21). The quotient set of $\breve{E}$ is the family $<\breve{E}>$ of $\mathcal{E}-$classes of interval ellipses. The fact that a quotient set $<\breve{E}>$ is not yet a vector space in the traditional sense is intriguing. Further, $\mathcal{E}([x,y],[u,v],a)\in<\mathcal{E}([x_1,y_1],[u_1,v_1],a)>\Rightarrow\mathcal{E}([x,y],[u,v],a)=\mathcal{E}([x_1,y_1],[u_1,v_1],a)$, $[x,y]\overset{\mathcal{N}}{=}[x_1,y_1]$ and $[u,v]\overset{\mathcal{N}}{=}[u_1,v_1]$. Equivalently, the family of equivalence $\mathcal{E}-$classes constitutes a partition of the entire set $\breve{E}$ of all $\mathcal{E}-$classes of interval ellipses defined on elements of $\mathcal{U}$.

Definition 44 *Let* $\mathcal{U}$ *be the set of closed and bounded intervals and* $\mathcal{N}$ *be the null set. Let* $\mathcal{E}([x,y],[u,v],\mathbf{a})$ *be an interval ellipse in a* $b-$*interval metric space* $(\mathcal{U},d_b)$, $s\geq1$ *and* $\mathcal{M}$ *be a self map defined on* $\mathcal{U}$. *An interval ellipse* $\mathcal{E}([x,y],[u,v],a)$ *is known as a near fixed interval ellipse of* $\mathcal{M}$ *iff* $\mathcal{M}[x,y]\overset{\mathcal{N}}{=}[x,y]$, $[x,y]\in\mathcal{E}([x,y],[u,v],a)$.

The near fixed interval ellipse of the function $\mathcal{M}$ and its equivalence class are then determined using a $b-$interval metric variation of the conventional Caristi map [3].

Theorem 45 *Let* $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ *be an interval ellipse in a* $b-$*interval metric space* $(\mathcal{U},d_b)$, $s\geq1$ *and the null set* $\mathcal{N}$. *Define* $\zeta:\mathcal{U}\to[0,\infty)$ *as:*

$$\zeta([x,y])=d_b([x,y],[x_0,y_0])+d_b([x,y],[u_0,v_0]),\ [x,y],[u,v]\in\mathcal{U} \tag{22}$$

If a self map $\mathcal{M} \quad \mathcal{U} \to \mathcal{U}$ satisfies

1. $d_b([x,y],\mathcal{M}[x,y]) \le \zeta([x,y]) - \zeta(\mathcal{M}[x,y])$;
2. $d_b(\mathcal{M}[x,y],[x_0,y_0]) + d_b(\mathcal{M}[x,y],[u_0,v_0]) \ge \mathbf{a}, \mathbf{a} \in [0,\infty)$.

Then, $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ is a near fixed interval ellipse of $\mathcal{M}$.

3. If

$$d_b(\mathcal{M}[x,y],\mathcal{M}[u,v]) \le \alpha d_b([x,y],[u,v]) + \beta d_b([x,y],\mathcal{M}[x,y]) + \gamma d_b([u,v],\mathcal{M}[u,v]) + \delta d_b([x,y],\mathcal{M}[u,v]) + \eta d_b([u,v],\mathcal{M}[x,y]), \tag{23}$$

$$[x,y],[u,v] \in \mathcal{U}, \text{ and } \alpha+\beta+\gamma+\delta+\eta < \frac{1}{\mathbf{s}}, \text{ then}$$

$$\mathcal{M}[x,y] \overset{\mathcal{N}}{=} [x,y] \Rightarrow \mathcal{M}[\bar{x},\bar{y}] \overset{\mathcal{N}}{=} [\bar{x},\bar{y}],\ [\bar{x},\bar{y}] \in <[x,y]>.$$

4. Further, if for $[x,y] \in \mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ and $[u,v] \in \mathcal{U} \setminus \mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$, contraction condition (23) is satisfied, then $\mathcal{M}$ has a unique equivalence $\mathcal{E}$ –class of near fixed interval ellipses

$<\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})>$, that is, if $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ is a near fixed interval ellipse of $\mathcal{M}$, then $\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{r}_0,\bar{s}_0],\mathbf{a}) \in <\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})>$ or
$<\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})> = <\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{r}_0,\bar{s}_0],\mathbf{a})>$.

Equivalently, if $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ and $\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{r}_0,\bar{s}_0],a)$ are the near fixed interval ellipses of $\mathcal{M}$, then $<\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})> = <\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{r}_0,\bar{s}_0],\mathbf{a})>$.

Proof. Let $[x,y] \in \mathcal{E}([x_0,y_0]],a)$ be any arbitrary point. Using condition 1 and equation (22)

$$\begin{aligned} d_b([x,y],\mathcal{M}\quad [x,y]) &\le \zeta([x,y]) - \zeta(\mathcal{M}[x,y]) \\ &= d_b([x,y],[x_0,y_0]) + d_b([x,y],[u_0,v_0]) - d_b(\mathcal{M}[x,y],[x_0,y_0]) - d_b(\mathcal{M}[x,y],[u_0,v_0]) \\ &= \mathbf{a} - d_b(\mathcal{M}[x,y],[x_0,y_0]) - d_b(\mathcal{M}[x,y],[u_0,v_0]) \\ &\le 0,\ (\textit{using } 2) \end{aligned}$$

and so $\mathcal{M}[x,y] \overset{\mathcal{N}}{=} [x,y]$, that is, $[x,y]$ is a near fixed point of $\mathcal{M}$. We assert that for point $[\bar{x},\bar{y}] \in <[x,y]>$, $\mathcal{M}[\bar{x},\bar{y}] \overset{\mathcal{N}}{=} [\bar{x},\bar{y}]$. Since, d_b satisfies null equalities, so $[\bar{x},\bar{y}] \oplus n_1 = [x,y] \oplus n_2,\ n_1,n_2 \in \mathcal{N}$. Now,

$$\begin{aligned} d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]) &= d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}] \oplus n_1) \\ &\le s[d_b(\mathcal{M}[\bar{x},\bar{y}],[x_n,y_n]) + d_b([x_n,y_n],[\bar{x},\bar{y}] \oplus n_1)] \\ &= s[d_b(\mathcal{M}[\bar{x},\bar{y}],\mathcal{M}[x_{n-1},y_{n-1}]) + d_b([x_n,y_n],[\bar{x},\bar{y}] \oplus n_1)] \\ &\le s[\alpha d_b([\bar{x},\bar{y}],[x_n,y_n]) + \beta d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]) + \gamma d_b([x_n,y_n],\mathcal{M}[x_n,y_n]) \\ &\quad + \delta d_b([\bar{x},\bar{y}],\mathcal{M}[x_n,y_n]) + \eta s d_b([x_n,y_n],[\mathcal{M}\bar{x},\mathcal{M}\bar{y}]) + d_b([x_n,y_n],[x,y] \oplus n_2)] \\ &= s[\alpha d_b([\bar{x},\bar{y}] \oplus n_1,[x_n,y_n]) + \beta d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]) + \gamma d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + \\ &\quad \delta d_b([\bar{x},\bar{y}] \oplus n_1,[x_{n+1},y_{n+1}]) + \eta s(d_b([x_n,y_n],[\bar{x},\bar{y}]) + d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}])) \\ &= s[\alpha d_b([x,y] \oplus n_2,[x_n,y_n]) + \beta d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]) + \gamma d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + \\ &\quad \delta d_b([x,y] \oplus n_2,[x_{n+1},y_{n+1}]) + \eta s(d_b([x_n,y_n],[\bar{x},\bar{y}] \oplus n_1) + d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}])) \\ &\quad + d_b([x_n,y_n],[x,y] \oplus n_2)] \\ &= s[\alpha d_b([x,y],[x_n,y_n]) + \beta d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}]) + \gamma d_b([x_n,y_n],[x_{n+1},y_{n+1}]) + \\ &\quad \delta d_b([x,y],[x_{n+1},y_{n+1}]) + \eta s(d_b([x_n,y_n],[x,y] \oplus n_2) + d_b([\bar{x},\bar{y}],\mathcal{M}[\bar{x},\bar{y}])) \\ &\quad + d_b([x_n,y_n],[x,y] \oplus n_2)] \\ &\to s(\beta + s\eta) d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}]),\ \ as\ \ n \to \infty. \end{aligned}$$

Therefore $(1-s\beta-s^2\eta)d_b(\mathcal{M}[\bar{x},\bar{y}],[\bar{x},\bar{y}])\to 0,\ \ as\ \ n\to\infty$,

that is, $\mathcal{M}[\bar{x},\bar{y}]\overset{\mathcal{N}}{=}[\bar{x},\bar{y}]$, for any $[\bar{x},\bar{y}]\in<[x,y]>$ and for all $[x,y]\in\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$,

which implies, $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ is a near fixed interval ellipse of $\mathcal{M}$.

If $[x_1,y_1]\in<[x_0,y_0]>$ and $[u_1,v_1]\in<[u_0,v_0]>$ then $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})=\mathcal{E}([x_1,y_1],[u_1,v_1],\mathbf{a})$,

that is, $\mathcal{E}([x_1,y_1],[u_1,v_1],\mathbf{a})\in<\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})>$.

Let there exists two equivalence classes of near fixed interval ellipses $<\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})>$ and $<\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{u}_0,\bar{v}_0],\mathbf{a})>$ of $\mathcal{M}$, that is, $\mathcal{M}$ satisfies conditions (1) and (2) for every near fixed interval ellipses $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ and $\mathcal{E}([\bar{x}_0,\bar{y}_0],[\bar{u}_0,\bar{v}_0],\mathbf{a})$, but $[\bar{x}_0,\bar{y}_0]\in<[x,y]>$ and $[\bar{u}_0,\bar{v}_0]\in<[u,v]>$.

Now, for $[x,y]\in\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ and $[u,v]\in\mathcal{U}\setminus\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$, $\mathcal{M}[x,y]\overset{\mathcal{N}}{=}[x,y]$ and $\mathcal{M}[u,v]\overset{\mathcal{N}}{=}[u,v]$.

Then, $\mathcal{M}[x,y]\oplus n_1=[x,y]\oplus n_2$ and $\mathcal{M}[u,v]\oplus n_3=[u,v]\oplus n_4$, for some $n_1,n_2,n_3,n_4\in\mathcal{N}$.

Now,

$$\begin{aligned}
d_b([x,y],[u,v]) &= d_b([x,y]\oplus n_2,[u,v]\oplus n_4)\\
&= d_b(\mathcal{M}[x,y]\oplus n_1,\mathcal{M}[u,v]\oplus n_3)\\
&= d_b(\mathcal{M}[x,y],\mathcal{M}[u,v])\\
&\le \alpha d_b([x,y],[u,v])+\beta d_b([x,y],\mathcal{M}[x,y])+\gamma d_b([u,v],\mathcal{M}[u,v])\\
&\quad +\delta d_b([x,y],\mathcal{M}[u,v])+\eta d_b([u,v],\mathcal{M}[x,y])\\
&= \alpha d_b([x,y],[u,v])+\beta d_b([x,y],[x,y])+\gamma d_b([u,v],[u,v])+\delta d_b([x,y],[u,v])\\
&\quad +\eta d_b([u,v],[x,y])\\
&= (\alpha+\delta+\eta)d_b([u,v],[x,y])\\
\Rightarrow (1-\alpha-\delta-\eta) &\ \ d_b([u,v],[x,y])\le 0,
\end{aligned}$$

a contradiction. Hence, $<\mathcal{E}([x_0,y_0],[u_0,v_0],a)>$ is a unique equivalence $\mathcal{E}-$class of a near fixed interval ellipses of $\mathcal{M}$.

Example 46 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a $b-$interval metric $d_b:\mathcal{U}\to\mathcal{U}$ be defined as $d_b([x,y],[u,v])=(x+y-u-v)^2$ with $s=2$. Let $[a_0,b_0]\in\mathcal{U}$ such that $d_b([-2,2],[a_0,b_0])+d_b([3,5],[a_0,b_0])>10$. The interval ellipse with foci $[-2,2],[3,5]$ and length of major axis 10 is given by*

$$\begin{aligned}
\mathcal{E}([-2,2],[3,5],10) &= \{[x,y]\in\mathcal{U}: d_b([x,y],[-2,2])+d_b([x,y],[3,5])=10\}\\
&= \{[x,y]\in\mathcal{U}: (x+y)^2+x+y-8)^2=10\}.
\end{aligned}$$

If a self map $\mathcal{M}:\mathcal{U}\to\mathcal{U}$ is $\mathcal{M}[x,y]=\begin{cases}[x,y], & [x,y]\in\mathcal{E}([-2,2],[3,5],10)\\ [a_0,\ b_0], & [x,y]\notin\mathcal{E}([-2,2],[3,5],10)\end{cases}$

and $d_b([x,y],[a_0,b_0])\le(\alpha+\beta+\gamma+\delta+\eta)d_b([x,y],[u,v])$, where, $\alpha+\beta+\gamma+\delta+\eta<\dfrac{1}{\mathbf{s}}$, $[x,y]\in\mathcal{E}([-2,2],[3,5],10)$ and $[u,v]\notin\mathcal{E}([-2,2],[3,5],10)$.

Then, $\mathcal{M}$ validates all the hypotheses of Theorem 45, that is, the set of near fixed points of $\mathcal{M}$, $\{[x,y]\in\mathcal{U}:(x+y)^2+(x+y-8)^2=10\}$ contains a near fixed interval ellipse $\mathcal{E}([-2,2],[3,5],10)$. However, one may notice that there are infinitely many near fixed interval ellipses contained in the unique equivalence $\mathcal{E}-$class $<\mathcal{E}([-2,2],[3,5],10)>$ of near fixed interval ellipses of $\mathcal{M}$.

A distinct equivalence $\mathcal{E}-$class of near fixed interval ellipses and the importance of conditions (1) and (2) in their continued existence are illustrated in the following examples.

Example 47 *Let $b-$interval metric be defined as in Example 46 and $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ be an interval ellipse defined on $\mathcal{U}$, whose eccentricity is less than $d_b([x_0,y_0],[u_0,v_0])$. Next, define a self map $\mathcal{M}:\mathcal{U}\to\mathcal{U}$ as $\mathcal{M}[x,y]=[x_0,y_0],[x,y]\in\mathcal{U}$.*

Then, map $\mathcal{M}$ verifies condition (1) but does not verify the conditions (2), (3), and (4) of Theorem 45. One may notice that $\mathcal{M}$ does not nearly fix the interval ellipse $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$.

Example 48 *Let $b-$interval metric be defined as in Example 46 and $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ be an interval ellipse defined on $\mathcal{U}$. Choose a point $[\alpha,\beta]\in\mathcal{U}$ such that $d_b([x_0,y_0],[a_0,b_0])+d_b([u_0,v_0],[a_0,b_0])=\rho>\mathbf{a}$. Next, define a self map $\mathcal{M}:\mathcal{U}\to\mathcal{U}$ as $\mathcal{M}[x,y]=[a_0,b_0],[x,y]\in\mathcal{U}$.*

Then, map $\mathcal{M}$ verifies the condition (2) but does not verify conditions (1), (3), and (4) of Theorem 45. One may notice that $\mathcal{M}$ does not nearly fix the interval ellipse $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$.

Theorem 49 *Theorem 45 still holds true even if we substitute* (1) *by* $(1)'$, (2) *by* $(2)'$.

$$2d_b([x,y],\mathcal{M}[x,y])\le\zeta([x,y])+\zeta(\mathcal{M}[x,y])-2\mathbf{a};$$
$$d_b(\mathcal{M}[x,y],[x_0,y_0])+d_b(\mathcal{M}[x,y],[u_0,v_0])\le\mathbf{a}.$$

Proof. Let $[x,y]\in\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ be any arbitrary point. Using $(1)'$ and equation (22)

$$\begin{aligned}2d_b([x,y],\mathcal{M}[x,y]) &\le d_b([x,y],[x_0,y_0])+d_b([x,y],[u_0,v_0])+d_b(\mathcal{M}[x,y],[x_0,y_0])\\ &\quad +d_b(\mathcal{M}[x,y],[u_0,v_0])-2\mathbf{a},\\ &=\mathbf{a}+d_b(\mathcal{M}[x,y],[x_0,y_0])+d_b(\mathcal{M}[x,y],[u_0,v_0])-2\mathbf{a}\\ &\le\mathbf{a}+\mathbf{a}-2\mathbf{a}=0,\ \textit{using}\ (2)',\end{aligned}$$

and so $\mathcal{M}[x,y]\overset{\mathcal{N}}{=}[x,y]$. Now, $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ is near fixed interval ellipse of $\mathcal{M}$ and the uniqueness of equivalence $\mathcal{E}-$class of near fixed interval ellipse of $\mathcal{M}$ may be concluded as in Theorem 45.

Example 50 *Let $\mathcal{U}$ be the set of closed and bounded intervals and $\mathcal{N}$ be the null set. Let a $b-$interval metric $d_b:\mathcal{U}\to\mathcal{U}$ be defined as $d_b([x,y],[u,v])=|x+y-u-v|^3$ with $s=4$. Choose, $[\alpha,\beta]\in\mathcal{U}$ such that $d_b([-8,1],[\alpha,\beta])+d_b([2,9],[\alpha,\beta])<20$. The interval ellipse*

$$\begin{aligned}\mathcal{E}([-8,1],[2,9],20) &=\{[x,y]\in\mathcal{U}:d_b([x,y],[-8,1])+d_b([x,y],[2,9])=20\}\\ &=\{[x,y]\in\mathcal{U}:|x+y+7|^3+|x+y-11|^3=20\}.\end{aligned}$$

Define a self map $\mathcal{M}:\mathcal{U}\to\mathcal{U}$ as $\mathcal{M}[x,y]=\begin{cases}[x,y], & [x,y]\in\mathcal{E}([-8,1],[2,9],20)\\ [a_0,b_0], & [x,y]\notin\mathcal{E}([-8,1],[2,9],20)\end{cases}$ and

$$d_b([x,y],[a_0,b_0])\le\frac{1}{250}d_b([x,y],[u,v]),\ \text{where,}\quad \alpha=\beta=\gamma=\delta=\eta<\frac{1}{1250},$$

$[x,y]\in\mathcal{E}([-8,1],[a_0,b_0],20)$ and $[u,v]\notin\mathcal{E}([-8,1],[2,9],20)$. Then, the self map $\mathcal{M}$ validates all the hypotheses $(1)'$, $(2)'$, (3) and (4) of Theorem 49 except (3), that is, the set of near fixed points of $\mathcal{M}$, $\{[x,y]\in\mathcal{U}:|x+y+7|^3+|x+y-11|^3=20\}$ contains a near fixed interval ellipse $\mathcal{E}([-8,1],[2,9],20)$. However, one may notice that there are infinitely many near fixed interval ellipses contained in the unique equivalence $\mathcal{E}-$class $<\mathcal{E}([-8,1],[2,9],20)>$ of near fixed interval ellipses of $\mathcal{M}$.

The following example depicts the significance of the conditions $(1)'$, $(2)'$, and (3) in the survival of a near fixed interval ellipse.

Example 51 *Let a $b-$interval metric be defined as in Example 50 and $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ be a near fixed interval ellipse defined on $\mathcal{U}$. Choose, a point $[\alpha,\beta]\in\mathcal{U}$ such that*

$$d_b([x_0,y_0],[a_0,b_0])+d_b([u_0,v_0],[a_0,b_0])=\rho<\mathbf{a}.$$

Next, define a self map $\mathcal{M}:\mathcal{U}\to\mathcal{U}$ as $\mathcal{M}[x,y]=[a_0,b_0],[x,y]\in\mathcal{U}$. Then, map $\mathcal{M}$ verifies the condition $(2)'$, but does not verify conditions $(1)'$, (3) and (4) of Theorem 49. One may notice that $\mathcal{M}$ does not nearly fix the interval ellipse $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$.

Remark 52

1. Since the circles are the ellipses of vanishing eccentricity in which both the focal points are the same, we may prove similar results for near fixed circles. Recently, Tomar et al. [31] initiated the work in this direction exploiting metric interval space.
2. It is not necessary that an interval ellipse is the same as an ellipse in a Euclidean space. Noticeably, interval ellipses discussed in Examples 42-51 are different from the ellipses in a Euclidean space.
3. Noticeably, the major axis a of the near fixed ellipse does not depend on the centre or foci and may not be maximal.
4. $\mathcal{M}\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a}) = \mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ does not imply that $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a})$ is a near fixed ellipse of $\mathcal{M}$. If both the focuses coincide, then interval fixed ellipse results reduce to analogous interval fixed circle (see, Tomar et al. [31] in metric interval space ($\mathbf{s}=1$)). Noticeably, if $[x_0,y_0]=[u_0,v_0]=[r_0,s_0]$ (say), $\mathcal{E}([x_0,y_0],[u_0,v_0],\mathbf{a}) = \mathcal{C}([r_0,s_0],\frac{\mathbf{a}}{2})$ with centre $[r_0,s_0]$ and radius $\frac{\mathbf{a}}{2}$.
5. It is worth mentioning here that Theorems 45 and 49 (see, supporting examples also) establish the significant fact that a discontinuous self-map may nearly fix an interval ellipse (that is, set of near fixed points of a discontinuous self map may include a near fixed interval ellipse) which naturally arise in numerous real-world problems.

5. Conclusion

In this manuscript, we utilised the Hardy-Rogers contraction in the context of the $b-$interval metric space, to identify the near fixed point and its distinct equivalency class. Examples 23-37 demonstrate the significant fact that a conventional Hardy-Rogers contraction principle [10] and improved quasi-contraction principle [5] may not be proved in a $b-$interval metric space deducing thereby that the celebrated outcomes in metric fixed point theory may not be proved in a novel $b-$interval metric space. However, Examples 23, 24, 36, and 37 demonstrated the significant fact that $b-$interval metric space has initiated an ambiance for the existence of a near fixed point and its unique equivalence class for continuous as well as discontinuous maps. As an application of the $b-$interval metric, we have studied the geometry of near fixed points, familiarizing the notion of a near fixed interval ellipse and its unique $\mathcal{E}-$class.

Availability of data and material

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Competing interests

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Authors' contributions

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