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Approximating to a common fixed point and a minimizer of a convex function in Hadamard spaces

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Abstract

This paper considers approximate sequences convergent to a common solution to a family of fixed point problems and convex minimization problems. We found that the lemma used to prove a known convergence theorem has a gap in its proof, and we obtained a counterexample. Further, we get an analogous result by substituting the convex combination of finitely many points by the balanced mapping.

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1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. The problem of finding a fixed point of T is one of the most important problems in nonlinear analysis. This problem has been studied by many researchers and applied to many problems.

Let f be a proper convex function of H into $]-\infty,\infty]$. The convex minimization problem is defined as to find a point $z_0 \in H$ satisfying

$$f(z_0) = \min_{y \in H} f(y).$$

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A resolvent operator of a convex function is important to solve this problem, which is defined by

$$J_{\lambda f}x = \operatorname*{Argmin}_{y\in H}\left\{f(y)+\frac{1}{2\lambda}\|x-y\|^2\right\}$$

for $x \in H$ and $\lambda > 0$; see details [1, 2, 11, 12].

A Hilbert space is a special case of a Banach space, and it is also a special case of a Hadamard space. In a Hadamard space X, a resolvent operator of a convex function is defined by

$$J_{\lambda f}x = \operatorname*{Argmin}_{y\in X}\left\{f(y)+\frac{1}{2\lambda}d(x,y)^2\right\}$$

for $x \in X$ and $\lambda > 0$; see [1] for more details. On the other hand, approximation methods for finding a fixed point of a nonexpansive mapping and a minimizer of a function in Hilbert spaces and Hadamard spaces are studied by many researchers; see [1, 7, 8].

In 2018, Lerkchaiyaphum and Phuengrattana [10] introduced a delta-convergence sequence generated by the convex combination of more than three points in a Hadamard space.

Theorem 1.1 (Lerkchaiyaphum and Phuengrattana): [10] *Let X be a Hadamard space, T_i a nonexpansive mapping of X into itself for $i \in \{1,2\ldots,N\}$, and f^i a proper, lower semicontinuous, and convex function of X into $[-\infty,\infty]$ for $i \in \{1,2\ldots,N\}$ such that $F = (\bigcap_{i=1}^N \mathrm{Fix}T_i) \cap (\bigcap_{i=1}^N \mathrm{Argmin}_X f^i) \neq \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\}, \{\beta_n^i \mid n \in \mathbb{N}, i \in \{0,1,\ldots,N\}\}, \{\gamma_n^i \mid n \in \mathbb{N}, i \in \{1,2,\ldots,N\}\} \subset [a,b] \subset]0,1[$ such that $\sum_{i=0}^N \beta_n^i = \sum_{i=0}^N \gamma_n^i = 1$ for $n \in \mathbb{N}$, and $\{\lambda_n^i \mid n \in \mathbb{N}, i \in \{1,2,\ldots,N\}\} \subset [0,\infty[$ such that $\lambda_n^i \to \lambda^i \in]0,\infty[$ for $i \in \{1,2,\ldots,N\}$. Sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by $x_1 \in X$ and*

$$\begin{aligned}
J_{\lambda_n^i f^i}x_n &= \operatorname*{Argmin}_{y\in X}\left\{f^i(y)+\frac{1}{2\lambda_n^i}d(x_n,y)^2\right\} \text{ for } i \in \{1,2,\ldots,\mathrm{N}\};\\
z_n &= \beta^0 x_n \oplus (1-\beta^0)\bigoplus_{i=1}^N \frac{\beta_n^i}{1-\beta^0} J_{\lambda_n^i f^i}x_n;\\
y_n &= \gamma_n^0 x_n \oplus (1-\gamma_n^0)\bigoplus_{i=1}^N \frac{\gamma_n^i}{1-\gamma_n^0} T_i x_n;\\
x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n) y_n
\end{aligned}$$

for each $n \in \mathbb{N}$, where for $\{\alpha^i \mid i \in \{1,2,\ldots,N\}\} \subset]0,1[$ with $\sum_{i=1}^N \alpha^i = 1$ and $\{y_1, y_2, \ldots, y_N\} \subset X$,

$$\bigoplus_{i=1}^N \alpha^i y_i = \begin{cases} y_1 & (N=1);\\ \alpha^N y_N \oplus (1-\alpha^N)\bigoplus_{i=1}^{N-1}\frac{\alpha^i}{1-\alpha^N}\alpha^i y_i & (N \geq 2).\end{cases}$$

Then, $\{x_n\}$ is delta-convergent to an element of F.

This theorem is proved using a lemma concerning an inequality about the convex combination of a finite number of points in a Hadamard space. However, the lemma has a gap in its proof, and we found a counterexample of the lemma; see Section 3.

In geodesic spaces, the convex combination of more than two points is order-dependent. In 2018, Hasegawa and Kimura [6] introduced another definition of convex combination of mappings which is order-independent for three or more points, which is called a balanced mapping.

In this paper, we introduce an iterative scheme analogous to that in Theorem 1.1 by substituting the convex combination of finitely many points by the balanced mapping. The iterative sequence is delta-convergent to a common fixed point and common minimizer of convex functions.

2. Preliminaries

Let (X,d) be a metric space and T a mapping of X into itself. The set of all fixed points of T is denoted by $\operatorname{Fix} T$. A mapping T is nonexpansive if for every $x,y \in X$, the inequality $d(Tx,Ty) \le d(x,y)$ holds. Let f be a function of X into $]-\infty,\infty]$. Then, the set of all minimizers of f is denoted by $\operatorname{Argmin}_{y\in X} f(y)$. Let $\{x_n\} \subset X$ be a bounded sequence. Then, a point $x_0 \in X$ is called and asymptotic center of $\{x_n\}$ if the equation

$$\limsup_{n\to\infty} d(x_n, x_0) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y)$$

holds. the set of all asymptotic centers of $\{x_n\}$ is denoted by $AC(\{x_n\})$. Further, we say a sequence $\{x_n\}$ is delta-convergent to x_0 if $AC(\{x_{n_i}\}) = \{x_0\}$ for all subsequence $\{x_{n_i}\} \subset \{x_n\}$, which is denoted it by $x_n \overset{\Delta}{\rightharpoonup} x_0$.

Let X be a metric space and $x,y \in X$. A mapping γ_{xy} of $[0,d(x,y)]$ into X is called a geodesic joining x and y if $\gamma(0) = x$, $\gamma(d(x,y)) = y$, and that $d(\gamma(s),\gamma(t)) = |s-t|$ for all $s,t \in [0,d(x,y)]$. A metric space X is called a uniquely geodesic space if for every $x,y \in X$, there exists a unique geodesic joining x and y. We denote the image of γ_{xy} by $\operatorname{Im}\gamma_{xy}$. Let X be a uniquely geodesic space. For each $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in X$ such that

$$d(x,z) = (1-t)d(x,y) \text{ and } d(y,z) = td(x,y),$$

which is denoted by $z = tx \oplus (1-t)y$. Such a point $z \in X$ is called a convex combination between x and y. For each $x,y,z \in X$, the set $\Delta(x,y,z) = \operatorname{Im}\gamma_{xy} \cup \operatorname{Im}\gamma_{yz} \cup \operatorname{Im}\gamma_{zx}$ is called a geodesic triangle. For each $\Delta(x,y,z) \subset X$, a comparison triangle of $\Delta(x,y,z)$ is defined by the set $\Delta(\bar{x},\bar{y},\bar{z}) \subset \mathbb{R}^2$ satisfying

$$d(x,y) = d(\bar{x},\bar{y}), d(y,z) = d(\bar{y},\bar{z}), \text{ and } d(z,x) = d(\bar{z},\bar{x}).$$

A point $\bar{p} \in \operatorname{Im}\gamma_{\bar{x}\bar{y}}$ is called a comparison point for $p \in \operatorname{Im}\gamma_{xy}$ if $d(x,p) = d(\bar{x},\bar{p})$. A uniquely geodesic space is called a CAT(0) space If for all $p,q \in \Delta(x,y,z)$ and their comparison points $\bar{p},\bar{q} \in \Delta(\bar{x},\bar{y},\bar{z}) \subset \mathbb{R}^2$, the inequality $d(p,q) \le d(\bar{p},\bar{q})$ holds for all triangles in X. A complete CAT(0) space is called a Hadamard space. In a Hadamard space, the following lemmas hold:

Lemma 2.1 (Baĉák, Bridson and Haefliger): [1] *Let X be a Hadamard space, $x,y,z \in X$ and $t \in [0,1]$. Then, the following inequality holds:*

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2.$$

Lemma 2.2 (Kirk and Panyanak): [9] *Let X be a Hadamard space. Then every bounded sequence has a subsequence which is delta-convergent to $x_0 \in X$.*

Lemma 2.3 (Dhompongsa, Kirk and Sims): [5] *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence of X. Then the asymptotic center of $\{x_n\}$ consists of one point.*

Let X be a uniquely geodesic space and f a function of X into $]-\infty,\infty]$. A function f is a proper if the set $\{x \in X \mid f(x) < \infty\}$ is nonempty. A function f is said to be *lower semicontinuous* if the set $\{x \in X \mid f(x) \le a\}$ is closed for all $a \in \mathbb{R}$. If f is continuous, then it is lower semicontinuous. A function f is said to be *convex* if

$$f(\alpha x \oplus (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

holds for all $x,y \in X$ and $\alpha \in [0,1]$.

In the following theorem, Baĉák [1] introduced a resolvent for convex function in a Hadamard space. Further, Kimura and Kohsaka [8] considered the properties this operator.

Theorem 2.1 (Bačák, Kimura and Kohsaka): [1] *Let X be a Hadamard space, and f a proper, lower semicontinuous and convex function of X into* $[-\infty,\infty]$. *Let*

$$J_{\eta f}x = \operatorname*{Argmin}_{y\in X}\left\{f(y)+\frac{1}{2\eta}d(x,y)^2\right\}$$

for $x\in X$ and $\eta>0$. Then the following conditions hold:

1. $J_{\eta f}$ is single-valued;
2. $\mathrm{Fix}J_{\eta f} = \mathrm{Argmin}_X f$;
3. $J_{\eta f}$ is nonexpansive;
4. The following inequality holds:

$$(\lambda+\mu)d(J_{\lambda f}x,J_{\mu f}y)^2+\mu d(J_{\lambda f}x,x)^2+\lambda d(J_{\mu f}y,y)^2 \\ \le \lambda d(J_{\lambda f}x,y)^2+\mu d(x,J_{\mu f}y)^2$$

for $x,y\in X$ and $\lambda,\mu>0$.

In 2018, Hasegawa and Kimura introduced a notion of a balanced mapping and consider its properties in Hadamard spaces.

Theorem 2.2 (Hasegawa and Kimura): [6] *Let X be a Hadamard space, T_i a nonexpansive mapping of X into itself for $i=1,2,\dots,N$ such that $\bigcap_{i=1}^{N}\mathrm{Fix}\,T_i\neq\emptyset$, and $\{\alpha^i : i=1,2,\dots,N\}\subset[0,1]$ such that $\sum_{i=1}^{N}\alpha^i=1$. Let*

$$Ux = \operatorname*{Argmin}_{y\in X}\sum_{i=1}^{N}\alpha^i d(T_i x,y)^2$$

for each $x\in X$. Then, the following hold:

1. U is single-valued;
2. U is nonexpansive;
3. $\mathrm{Fix}U=\bigcap_{i=1}^{N}\mathrm{Fix}T_i$.

Remark 1: *Let $\{x_1,x_2,\dots,x_N\}$ be points in a Hilbert space. Then, for $\{\alpha_i \mid i\in\{1,2,\dots,N\}\}\subset\,]0,1[$ such that $\sum_{i=1}^{N}\alpha_i=1$, we know that*

$$\operatorname*{Argmin}_{y\in H}\sum_{i=1}^{N}\alpha_i\|x_i-y\|^2=\sum_{i=1}^{N}\alpha_i x_i.$$

Therefore, the mapping U in the theorem above is a usual convex combination of mappings $\{T_i\}$ if X is a Hilbert space.

3. Inequality of convex combination and its counterexample

Let X be a Hadamard space. In 2014, Chidume et.al. [4] claimed the following inequality:

$$d\left(\bigoplus_{i=1}^{N}\alpha^i x_i,z\right)^2\le\sum_{i=1}^{N}\alpha^i d(x_i,z)^2-\sum_{i=2}^{N}\sum_{j=1}^{i-1}\alpha^i\alpha^j d(x_i,x_j)^2 \tag{1}$$

for $z\in X$, $\{x_1,x_2,\dots,x_N\}\subset X$, and $\{\alpha^i \mid i\in\{1,2,\dots,N\}\}\subset]0,1[$ with $\sum_{i=1}^{N}\alpha^i=1$. However, this inequality does not hold in general. Indeed, let X be a uniquely geodesic space and $Y\subset X$ satisfying $Y=[x_1,z]\cup[x_2,z]\cup[x_3,z]$ for $x_1,x_2,x_3,z\in X$ with

$$d = d(x_1, z) = d(x_2, z) = d(x_3, z).$$

Then, Y is an $\mathbb{R}$-tree, and thus it is a complete CAT(0) space [1, 3]. We also have

$$d(x_1, x_2) = d(x_1, z) + d(z, x_2) = 2d \text{and} d(x_2, x_3) = d(x_3, x_1) = 2d.$$

Let $\alpha_i = 1/3$ for $i \in \{1,2,3\}$. Then

$$\bigoplus_{i=1}^{3} \alpha_i x_i = \bigoplus_{i=1}^{3} \frac{1}{3} x_i = \frac{1}{3} x_3 \oplus \frac{2}{3}\left(\frac{1}{2} x_2 \oplus \frac{1}{2} x_1\right) = \frac{1}{3} x_3 \oplus \frac{2}{3} z,$$

and thus

$$d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, z\right)^2 = d\left(\bigoplus_{i=1}^{3} \frac{1}{3} x_i, z\right)^2 = d\left(\frac{1}{3} x_3 \oplus \frac{2}{3} z, z\right)^2 = \frac{1}{9} d(x_3, z)^2 = \frac{1}{9} d^2.$$

On the other hand, we have

$$\begin{aligned}
&\sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i=2}^{3}\sum_{j=1}^{i-1} \alpha_i \alpha_j d(x_i, x_j)^2 \\
&= \frac{1}{3} d(x_1, z)^2 + \frac{1}{3} d(x_2, z)^2 + \frac{1}{3} d(x_3, z)^2 - \frac{1}{9} d(x_1, x_2)^2 - \frac{1}{9} d(x_2, x_3)^2 - \frac{1}{9} d(x_3, x_1)^2 \\
&= \frac{1}{3} d^2 - \frac{4}{9} d^2 - \frac{4}{9} d^2 - \frac{4}{9} d^2 \\
&= -\frac{1}{3} d^2
\end{aligned}$$

and hence

$$d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, z\right)^2 > \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i=2}^{3}\sum_{j=1}^{i-1} \alpha_i \alpha_j d(x_i, x_j)^2.$$

It is a counterexample to the inequality (1).

4. Approximation theorem using a balanced mapping

In this section, we introduce an iterative scheme similar to that is Theorem 1.1 using a balanced mapping of a finite family of resolvent operators and nonexpansive mappings. We prove its delta-convergence to a common fixed point of nonexpansive mappings and minimizer of convex functions in a Hadamard space.

Theorem 4.1: *Let X be a Hadamard space, T_i a nonexpansive mapping of X into itself for $i \in \{1,2,\dots,N\}$, and f^i a proper, lower semicontinuous, and convex function of X into $]-\infty,\infty]$ for $i \in \{1,2,\dots,N\}$ such that $F = (\bigcap_{i=1}^{N} \mathrm{Fix}\, T_i) \cap (\bigcap_{i=1}^{N} \mathrm{Argmin}_X f^i) \neq \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\}, \{\beta_n^i \mid n \in \mathbb{N}, i \in \{0,1,\dots,N\}\}, \{\gamma_n^i \mid n \in \mathbb{N}, i \in \{0,1,\dots,N\}\} \subset [a,b] \subset]0,1[$ such that $\sum_{i=0}^{N} \beta_n^i = \sum_{i=0}^{N} \gamma_n^i = 1$ for $n \in \mathbb{N}$, and $\{\lambda_n^i \mid n \in \mathbb{N}, i \in \{1,2,\dots,N\}\} \subset [0,\infty]$ such that $\lambda_n^i \to \lambda^i \in]0,\infty[$ for $i \in \{1,2,\dots,N\}$. Sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by $x_1 \in X$ and*

$$\begin{aligned} J_{\lambda_n^i f^i} x_n &= \operatorname*{Argmin}_{y\in X}\left\{ f^i(y) + \frac{1}{2\lambda_n^i} d(x_n, y)^2 \right\} \text{ for } i \in \{1,2,\dots,N\};\\ z_n &= \operatorname*{Argmin}_{y\in X}\left\{ \beta_n^0 d(x_n, y)^2 + \sum_{i=1}^N \beta_n^i d(J_{\lambda_n^i f^i} x_n, y)^2 \right\};\\ y_n &= \operatorname*{Argmin}_{y\in X}\left\{ \gamma_n^0 d(z_n, y)^2 + \sum_{i=1}^N \gamma_n^i d(T_i z_n, y)^2 \right\};\\ x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n) y_n \end{aligned}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ is delta-convergent to an element of F.

Proof. Let $p \in F$. Put

$$U_n x = \operatorname*{Argmin}_{y\in X}\left\{ \beta_n^0 d(x, y)^2 + \sum_{i=1}^N \beta_n^i d(J_{\lambda_n^i f^i} x, y)^2 \right\}$$

and

$$V_n x = \operatorname*{Argmin}_{y\in X}\left\{ \gamma_n^0 d(x, y)^2 + \sum_{i=1}^N \gamma_n^i d(T_i x, y)^2 \right\}$$

for $n \in \mathbb{N}$ and $x \in X$. Then, we have $z_n = U_n x_n$ and $y_n = V_n z_n$. Since fixed points of the identity mapping is the whole space and it is nonexpansive, by Theorem 2.2, it follows that $F = (\operatorname{Fix} U_n) \cap (\operatorname{Fix} V_n)$ for each $n \in \mathbb{N}$. Since U_n and V_n are nonexpansive for $n \in \mathbb{N}$, we get

$$d(z_n, p) = d(U_n x_n, p) \le d(x_n, p)$$

and

$$d(y_n, p) = d(V_n z_n, p) \le d(z_n, p) = d(U_n x_n, p) \le d(x_n, p)$$

for $n \in \mathbb{N}$. Then, we get

$$d(x_{n+1}, p) \le \alpha_n d(x_n, p) + (1-\alpha_n) d(y_n, p) \le d(x_n, p).$$

Hence $\{x_n\}$ is bounded, and $\{d(x_n, p)\}$ is nonincreasing. Put $c_p = \lim_{n\to\infty} d(x_n, p)$. By Lemma 2.1, it follows that

$$\begin{aligned} d(x_{n+1}, p)^2 &\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(y_n, p)^2 - \alpha_n(1-\alpha_n) d(x_n, y_n)^2\\ &\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(x_n, p)^2 - \alpha_n(1-\alpha_n) d(x_n, y_n)^2\\ &\le d(x_n, p)^2 - \alpha_n(1-\alpha_n) d(x_n, y_n)^2 \end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n) d(x_n, y_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $0 < a \le \alpha_n \le b < 1$ for $n \in \mathbb{N}$, we have

$$0 \le a(1-b) d(x_n, y_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Letting $n \to \infty$, we have

$$0 \le a(1-b) \lim_{n\to\infty} d(x_n, y_n)^2 \le c_p^2 - c_p^2 = 0$$

and hence $\lim_{n\to\infty} d(x_n, y_n) = 0$. Further, we obtain

$$\begin{aligned} c_p &= \lim_{n\to\infty} d(x_n, p) \\ &= \liminf_{n\to\infty} d(x_n, p) \\ &\le \liminf_{n\to\infty} (d(x_n, y_n) + d(y_n, p)) \\ &= \liminf_{n\to\infty} d(y_n, p) \\ &\le \limsup_{n\to\infty} d(y_n, p) \\ &\le \limsup_{n\to\infty} d(x_n, p) \\ &= \lim_{n\to\infty} d(x_n, p) = c_p \end{aligned}$$

and thus

$$c_p \le \liminf_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le c_p.$$

This implies that $\lim_{n\to\infty} d(y_n, p) = c_p$. Let $t \in [0,1]$ and put $\tau = tz_n \oplus (1-t)p$. By the definition of $\{z_n\}$, it follows that

$$\begin{aligned} \beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 &= \beta_n^0 d(U_n x_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(U_n x_n, J_{\lambda_n^i f^i} x_n)^2 \\ &\le \beta_n^0 d(\tau, x_n)^2 + \sum_{i=1}^N \beta_n^i d(\tau, J_{\lambda_n^i f^i} x_n)^2 \\ &\le \beta_n^0 (t d(z_n, x_n)^2 + (1-t) d(p, x_n)^2 - t(1-t) d(z_n, p)^2) \\ &\quad + \sum_{i=1}^N \beta_n^i (t d(z_n, J_{\lambda_n^i f^i} x_n)^2 + (1-t) d(p, J_{\lambda_n^i f^i} x_n)^2 - t(1-t) d(z_n, p)^2) \\ &= t\left(\beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \right) \\ &\quad + (1-t)\left(\beta_n^0 d(p, x_n)^2 + \sum_{i=1}^N \beta_n^i d(p, J_{\lambda_n^i f^i} x_n)^2 \right) - t(1-t) d(z_n, p)^2 \\ &\le t\left(\beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \right) \\ &\quad + (1-t)\left(\beta_n^0 d(p, x_n)^2 + \sum_{i=1}^N \beta_n^i d(p, x_n)^2 \right) - t(1-t) d(z_n, p)^2 \\ &\le t\left(\beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \right) + (1-t) d(x_n, p)^2 - t(1-t) d(z_n, p)^2 \end{aligned}$$

and hence

$$(1-t)\left(\beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^N \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \right) \le (1-t) d(x_n, p)^2 - t(1-t) d(z_n, p)^2.$$

Dividing by $1-t > 0$ and letting $t \searrow 0$, we get

$$\begin{aligned}\beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^{N} \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 &\le d(x_n, p)^2 - d(z_n, p)^2 \\ &\le d(x_n, p)^2 - d(y_n, p)^2.\end{aligned}$$

Since $0 < a \le \beta_n^i \le b < 1$ for $n \in \mathbb{N}$ and $i \in \{1, 2 \ldots, N\}$, we get

$$\begin{aligned}0 \le ad(z_n, x_n)^2 &\le \beta_n^0 d(z_n, x_n)^2 \\ &\le \beta_n^0 d(z_n, x_n)^2 + \sum_{i=1}^{N} \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \\ &\le d(x_n, p)^2 - d(y_n, p)^2.\end{aligned}$$

Letting $n \to \infty$, we get

$$0 \le a \lim_{n\to\infty} d(z_n, x_n)^2 \le c_p^2 - c_p^2 = 0$$

abd hence $\lim_{n\to\infty} d(x_n, z_n) = 0$. Fix $j \in \{1, 2, \ldots, N\}$ arbitrarily. Then,

$$\begin{aligned}0 \le ad(z_n, J_{\lambda_n^j f^j} x_n)^2 &\le \beta_n^j d(z_n, J_{\lambda_n^j f^j} x_n) \\ &\le \sum_{i=1}^{N} \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \\ &\le \beta_n^0 d(x_n, z_n)^2 + \sum_{i=1}^{N} \beta_n^i d(z_n, J_{\lambda_n^i f^i} x_n)^2 \\ &\le d(x_n, p)^2 - d(y_n, p)^2.\end{aligned}$$

Letting $n \to \infty$, we have

$$0 \le a \lim_{n\to\infty} d(z_n, J_{\lambda_n^j f^j} x_n)^2 \le c_p^2 - c_p^2 = 0$$

and thus $\lim_{n\to\infty} d(z_n, J_{\lambda_n^j f^j} x_n) = 0$. Then, it follows that

$$d(x_n, J_{\lambda_n^j f^j} x_n) \le d(x_n, z_n) + d(z_n, J_{\lambda_n^j f^j} x_n).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, J_{\lambda_n^j f^j} x_n) = 0$. By Lemma 2.1, we have

$$\begin{aligned}&(\lambda_n^j + \lambda^j) d(J_{\lambda_n^j f^j} x_n, J_{\lambda^j f^j} x_n)^2 + \lambda^j d(J_{\lambda_n^j f^j} x_n, x_n)^2 + \lambda_n^j d(J_{\lambda^j f^j} x_n, x_n)^2 \\ &\quad \le \lambda_n^j d(J_{\lambda_n^j f^j} x_n, x_n)^2 + \lambda^j d(x_n, J_{\lambda^j f^j} x_n)^2\end{aligned}$$

and hence

$$\begin{aligned}&(\lambda_n^j + \lambda^j) d(J_{\lambda_n^j f^j} x_n, J_{\lambda^j f^j} x_n)^2 \\ &\quad \le (\lambda_n^j - \lambda^j) d(J_{\lambda_n^j f^j} x_n, x_n)^2 + (\lambda_n^j - \lambda^j) d(J_{\lambda^j f^j} x_n, x_n)^2 \\ &\quad \le (\lambda_n^j - \lambda^j) d(J_{\lambda_n^j f^j} x_n, x_n)^2 + (\lambda_n^j - \lambda^j)(d(J_{\lambda^j f^j} x_n, p) + d(p, x_n))^2 \\ &\quad \le (\lambda_n^j - \lambda^j) d(J_{\lambda_n^j f^j} x_n, x_n) + 4(\lambda_n^j - \lambda^j) d(x_n, p)^2.\end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(J_{\lambda_n^j f^j} x_n, J_{\lambda^j f^j} x_n) = 0$. Then, we get

$$d(x_n, J_{\lambda^j f^j} x_n) \le d(x_n, J_{\lambda_n^j f^j} x_n) + d(J_{\lambda_n^j f^j} x_n, J_{\lambda^j f^j} x_n).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} d(x_n, J_{\lambda^j f^j} x_n) = 0.$$

Put $\tau' = t y_n \oplus (1-t)p$. By the definition of $\{y_n\}$, it follows that

$$\begin{aligned}
\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 &= \gamma_n^0 d(V_n z_n, x_n)^2 + \sum_{i=1}^N \gamma_n^i d(V_n z_n, T_i z_n)^2 \\
&\le \gamma_n^0 d(\tau', z_n)^2 + \sum_{i=1}^N \gamma_n^i d(\tau', T_i z_n)^2 \\
&\le \gamma_n^0 (t d(y_n, z_n)^2 + (1-t) d(p, z_n)^2 - t(1-t) d(y_n, p)^2) \\
&\quad + \sum_{i=1}^N \gamma_n^i (t d(y_n, T_i z_n)^2 + (1-t) d(p, T_i z_n)^2 - t(1-t) d(y_n, p)^2) \\
&= t\left(\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \right) \\
&\quad + (1-t)\left(\gamma_n^0 d(p, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(p, T_i z_n)^2 \right) - t(1-t) d(y_n, p)^2 \\
&\le t\left(\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \right) \\
&\quad + (1-t)\left(\gamma_n^0 d(p, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(p, z_n)^2 \right) - t(1-t) d(y_n, p)^2 \\
&\le t\left(\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \right) + (1-t) d(z_n, p)^2 - t(1-t) d(y_n, p)^2
\end{aligned}$$

and hence

$$(1-t)\left(\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \right) \le (1-t) d(z_n, p)^2 - t(1-t) d(y_n, p)^2.$$

Dividing by $1-t > 0$ and letting $t \searrow 0$, we get

$$\gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \le d(z_n, p)^2 - d(y_n, p)^2 \le d(x_n, p)^2 - d(y_n, p)^2.$$

Since $0 < a \le \gamma_n^i \le b < 1$ for $n \in \mathbb{N}$ and $i \in \{1, 2 \ldots, N\}$, we get

$$\begin{aligned}
a d(y_n, z_n)^2 &\le \gamma_n^0 d(y_n, z_n)^2 \\
&\le \gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \\
&\le d(x_n, p)^2 - d(y_n, p)^2.
\end{aligned}$$

Letting $n \to \infty$,

$$0 \le a \lim_{n\to\infty} d(y_n, z_n)^2 \le c_p^2 - c_p^2 = 0$$

and hence $\lim_{n\to\infty} d(y_n, z_n) = 0$. Fix $j' \in \{1, 2, \ldots, N\}$ arbitrarily. Then,

$$\begin{aligned} ad(y_n, T_{j'}z_n)^2 &\le \gamma_n^{j'} d(y_n, T_{j'}z_n)^2 \\ &\le \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \\ &\le \gamma_n^0 d(y_n, z_n)^2 + \sum_{i=1}^N \gamma_n^i d(y_n, T_i z_n)^2 \\ &\le d(x_n, p)^2 - d(y_n, p)^2. \end{aligned}$$

Letting $n \to \infty$, we have

$$0 \le a \lim_{n\to\infty} d(y_n, T_{j'}z_n)^2 \le c_p^2 - c_p^2 = 0$$

and thus $\lim_{n\to\infty} d(y_n, T_{j'}z_n) = 0$. Then, it follows that

$$d(z_n, T_{j'}z_n) \le d(z_n, y_n) + d(y_n, T_{j'}z_n).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(z_n, T_{j'}z_n) = 0$. Further, we have

$$\begin{aligned} d(x_n, T_{j'}x_n) &\le d(x_n, z_n) + d(z_n, T_{j'}z_n) + d(T_{j'}z_n, T_{j'}x_n) \\ &\le d(x_n, z_n) + d(z_n, T_{j'}z_n) + d(z_n, x_n) \to 0 \end{aligned}$$

as $n \to \infty$. Therefore we get

$$\lim_{n\to\infty} d(x_n, T_{j'}x_n) = 0.$$

Take $\{x_{n_k}\} \subset \{x_n\}$ arbitrarily. Since $\{x_{n_k}\}$ is bounded and Lemma 2.2, there exists $\{x_{n_{k_l}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_l}} \overset{\Delta}{\rightharpoonup} x_0 \in X$. Since $J_{\lambda^j f^j}$ is nonexpansive, we have

$$\begin{aligned} \limsup_{l\to\infty} d(x_{n_{k_l}}, J_{\lambda^j f^j} x_0) &\le \limsup_{l\to\infty} \left(d(x_{n_{k_l}}, J_{\lambda^j f^j} x_{n_{k_l}}) + d(J_{\lambda^j f^j} x_{n_{k_l}}, J_{\lambda^j f^j} x_0) \right) \\ &= \limsup_{l\to\infty} d(J_{\lambda^j f^j} x_{n_{k_l}}, J_{\lambda^j f^j} x_0) \\ &\le \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0). \end{aligned}$$

By Lemma 2.3, we get $J_{\lambda^j f^j} x_0 = x_0$. Since $j \in \{1, 2, \ldots, N\}$ is arbitrary, we get $x_0 \in \bigcap_{i=1}^N \mathrm{Argmin}_X f^i$. By the nonexpansiveness of $T_{j'}$, we get

$$\begin{aligned} \limsup_{l\to\infty} d(x_{n_{k_l}}, T_{j'}x_0) &\le \limsup_{l\to\infty} \left(d(x_{n_{k_l}}, T_{j'}x_{n_{k_l}}) + d(T_{j'}x_{n_{k_l}}, T_{j'}x_0) \right) \\ &= \limsup_{l\to\infty} d(T_{j'}x_{n_{k_l}}, T_{j'}x_0) \\ &\le \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0). \end{aligned}$$

By Lemma 2.3, we have $x_0 = T_{j'}x_0$. Since $j' \in \{1,2\ldots,N\}$ is arbitrary, we have $x_0 \in \bigcap_{i=1}^N \mathrm{Fix}\, T_i$ and hence $x_0 \in F$. Since $\{x_n\}$ is bounded, we can put $AC(\{x_n\}) = \{x_0'\}$ and $AC(\{x_{n_k}\}) = \{x_0''\}$. Then, we get

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, x_0) &= \lim_{l\to\infty} d(x_{n_{k_l}}, x_0) \\ &= \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0) \\ &\le \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0'') \\ &\le \limsup_{k\to\infty} d(x_{n_k}, x_0'') \\ &\le \limsup_{k\to\infty} d(x_{n_k}, x_0') \\ &\le \limsup_{n\to\infty} d(x_n, x_0') \le \limsup_{n\to\infty} d(x_n, x_0)\end{aligned}$$

and hence $x_0 = x_0' = x_0''$. Therefore, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in F$ and complete the proof.

By Remark 1 and Theorem 4.1, we can prove the following corollary in a Hilbert space.

Corollary 4.1: *Let H be a Hilbert space and C a nonempty closed and convex subset. Let T_i be a nonexpansive mapping of C into itself for $i \in \{1,2,,\ldots,N\}$, and f^i a proper, lower semicontinuous, and convex function of C into $]-\infty,\infty]$ such that $F = (\bigcap_{i=1}^N \mathrm{Fix} T_i) \cap (\bigcap_{i=1}^N \mathrm{Argmin}_X f^i) \neq \varnothing$. Let $\{\alpha_n \mid n \in \mathbb{N}\}, \{\beta_n^i \mid n \in \mathbb{N}, i \in \{0,1,\ldots,N\}\}, \{\gamma_n^i \mid n \in \mathbb{N}, i \in \{0,1,2,\ldots,N\}\} \subset [a,b] \subset [0,1]$ such that $\sum_{i=0}^N \beta_n^i = \sum_{i=0}^N \gamma_n^i = 1$ for $n \in \mathbb{N}$, and $\{\lambda_n^i \mid n \in \mathbb{N}, i \in \{1,2,\ldots,N\}\} \subset]0,\infty[$ such that $\lambda_n^i \to \lambda^i \in]0,\infty[$ for $i \in \{1,2,\ldots,N\}$. Sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by $x_1 \in C$ and*

$$\begin{aligned}J_{\lambda_n^i f^i} x_n &= \operatorname*{Argmin}_{y\in X}\left\{ f^i(y) + \frac{1}{2\lambda_n^i}\| x_n - y\|^2 \right\} \text{for } i \in \{1,2,\ldots,N\}; \\ z_n &= \beta_n^0 x_n + \sum_{i=1}^N \beta_n^i J_{\lambda_n^i f^i} x_n; \\ y_n &= \gamma_n^0 z_n + \sum_{i=1}^N \gamma_n^i T_i z_n; \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) y_n\end{aligned}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ is weakly convergent to an element of F.

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