Results in Nonlinear Analysis 8 (2025) No. 3, 210–221
https://doi.org/10.31838/rna/2023.08.03.017
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

A study of bi-convex classes in Leaf-like domains using quantum calculus through subordination

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Abstract

This research explores properties of a bi-convex class of functions that are associated with a leaf-shaped region by utilizes the subordination principle and q-calculus. The study also analyzes limitations on coefficients, with a particular emphasis on $|a_2|$ and $|a_3|$. Furthermore, it evaluates Fekete Szegö inequalities for functions within the bi-convex class. The findings are supported by figures, examples, and references to relevant studies.

Mathematics Subject Classification (2020): 30Cxx

Key words and Phrases: q-calculus leaf-like domain analytic functions bi-univalent functions convex class subordination Fekete-Szegö problem.

1. Introduction and Definitions

Let f be a holomorphic function defined within the open unit disc $\mathbb{U}$, where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function can be categorised as belonging to a specific class $\mathcal{A}$ if it can be expressed in the following manner:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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Received June 16, 2025; Accepted July 26, 2025; Online December 31, 2025

Furthermore included in the collection $\mathcal{S}$ are univalent functions satisfying the normalising requirements:

$f(0) = 0$ and $f'(0) = 1$. (2)

An analysis function $\varkappa$ defined within the domain $\mathbb{U}$ is referred to as a Schwarz function if it meets the conditions $\varkappa(z) < 1$ and $\varkappa(0) = 0$. In the context of a class $\mathcal{A}$ comprising two functions, f and f it is stated that f is influenced by f denoted as $f \prec f$, when there exists a Schwarz function $\varkappa$ that $\varkappa(z) = f(\varkappa(z))$, for every complex number $(z \in \mathbb{U})$.

The class P is closely related to Carathéodory functions, which are described by Miller [27]. These functions are characterized by the following features:

$$\mathrm{Re}\{\delta(z)\} > 0, \quad \text{and} \quad \delta(0) = 1 \qquad \forall\ z \in \mathbb{U}.$$

A Maclaurin series expansion provides a way to represent a function $\delta(z)$, within the class P. It can be written as;

$$\delta(z) = 1 + \sum_{n=1}^{\infty} \delta_n\ z^n, \quad (z \in \mathbb{U}). \tag{3}$$

In this expression the coefficients δ_n must satisfy the condition;

$$|\,\delta_n\,| \le 2, \quad \text{for all} \quad n \ge 1. \tag{4}$$

This concept aligns with Carathodorys Lemma, a known result discussed in the reference by Duren [20]. Essentially a function δ belongs to the class P if and only if it satisfies the condition;

$$\delta(z) \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{U}).$$

For any function f in the category $\mathcal{S}$, an inverse function denoted by f^1 exists. The inverse function is defined as follows:

$$z = f^{-1}(f(z)), \quad \varsigma = f(f^{-1}(\varsigma)) \qquad \left(r_0(f) \ge \frac{1}{4};\ |\varsigma| < r_0(f); z \in \mathbb{U} \right),$$

where

$$f^{-1}(\varsigma) = g(\varsigma) = \varsigma - \varrho_2\varsigma^2 + (2a_2^2 - a_3)\varsigma^3 - (a_4 + 5a_2^3 - 5a_3a_2)\varsigma^4 + \cdots. \tag{5}$$

A function $f(z) \in \mathcal{S}$ is called bi-univalent when its inverse function $(f^{-1}(\varsigma))$ is also bi-univalent. The set Σ includes all bi-univalent functions, in $\mathbb{U}$. Check out the table below for some examples of functions, in the class Σ and their respective inverse functions.

Table 1: Certain univalent functions along with their inverses.

$f_1(z) = \dfrac{z}{z+1}$	$f_1^{-1}(\varsigma) = \dfrac{\varsigma}{1-\varsigma}$
$f_2(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$	$f_2^{-1}(\varsigma) = -\log(1-\varsigma)$
$f_3(z) = \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$	$f_3^{-1}(\varsigma) = \dfrac{e^{\varsigma}-1}{e^{\varsigma}}$

Ma and Minda [26] developed the class $K(\aleph)$ by use of subordination. They put up their method as follows:

$$K(\aleph)=\left\{f\in\mathcal{A}:\ \ 1+\frac{zf''(z)}{f'(z)}\prec\aleph(z),\qquad (z\in\mathbb{U})\right\}.$$

In this formula, $\aleph(z)$ denotes a function with a positive real component and is normalised in accordance with criteria (2). The table below illustrates the diverse methodologies employed by several writers to delineate subclasses of functions by selecting particular forms for $\aleph$.

Table 2: Certain categories of functions are characterized by subordination.

$\aleph(z)$	**Author/s**	**Reference**	**Year**
$\sqrt{1+z}$	Sokol and Stankiewicz	[33]	1996
$\dfrac{z}{1-z}$	Piejko and Sokól	[29]	2012
$z+\sqrt{1+z^2}$	Priya and Sharma	[30]	2018
$z+\sqrt[3]{1+z^3}$	Singh and Kaur	[32]	2021

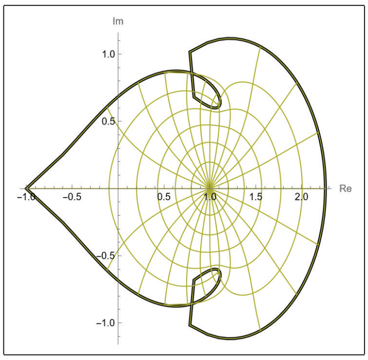


Figure 1: The image of $\aleph(\mathbb{U})$ is displayed in a leaf-shaped region. Here, $\aleph(z;1)=z+\sqrt[3]{1+z^3}$.

Quantum calculus, also known as q-calculus, extends beyond the conventional framework of ordinary calculus by incorporating the parameter $q\in(0,1)$, thereby generalizing classical analytical techniques. This field has garnered significant interest due to its deep connections with physics, quantum mechanics, and Geometric Function Theory (GFT). A foundational resource for understanding q-difference calculus and its diverse applications is the work of Gasper and Rahman [24], which provides a comprehensive exposition on the subject. Central to the study of analytic functions within this framework is the q-difference operator ∂_q, which plays a crucial role in function theory. Notable advancements in this area include the work of Seoudy and Aouf [31], who extended q-calculus to functions within the unit disc, further enriching GFT. For further exploration, numerous classical and contemporary studies provide valuable insights, including [1, 3, 6–13, 17, 18, 25, 28, 35].

Recently, Alsoboh and Oros [7] investigated a particular class of bi-univalent functions connected to the leaf-like domain associated with q-calculus. The study concentrated on the definition that follows:

Definition 1.1: *A function $f\in\mathcal{S}$ is considered a member of the class $\Sigma_q(\lambda,\aleph(z;q))$ if it has the form (1) and satisfies the following subordinations:*

$$(1-\lambda)\frac{f(z)}{z}+\lambda\left(\partial_q f(z)\right)\prec\aleph(z;q),\quad (z\in\mathbb{U}),\tag{6}$$

where $\aleph(0;q)=1,\ \lambda\geq 1$, *and*

$$\aleph(z;q)=\frac{(1+q)z}{2+(1-q)z}+\sqrt[3]{1+\left(\frac{(1+q)z}{2+(1-q)z}\right)^{3}}, \tag{7}$$

where

$$\partial_q f(z)=\begin{cases}\dfrac{f(z)-f(qz)}{z-qz}, & if\ z\neq 0,\\ f'(z), & if\ q\to 1^-,\ z\neq 0,\\ f'(0) & if\ z=0\end{cases}$$

Recent works by Amourah et al. [14] established the starlike class $\mathcal{S}^*(q;\aleph(z;q))$, which pertains to the leaf-like domain within $\mathbb{U}$. This class is defined as follows:

Definition 1.2: *A function $f\in\mathcal{S}$ is considered a member of the class $\mathcal{S}^*(q;\aleph(z;q))$ if it has the form (1) and satisfies the following subordinations:*

$$\frac{z\partial_q f(z)}{\psi(z)}\prec\aleph(z;\,q),\qquad (z\in\mathbb{U}), \tag{8}$$

where $\aleph(z;q)$ of the form (7).

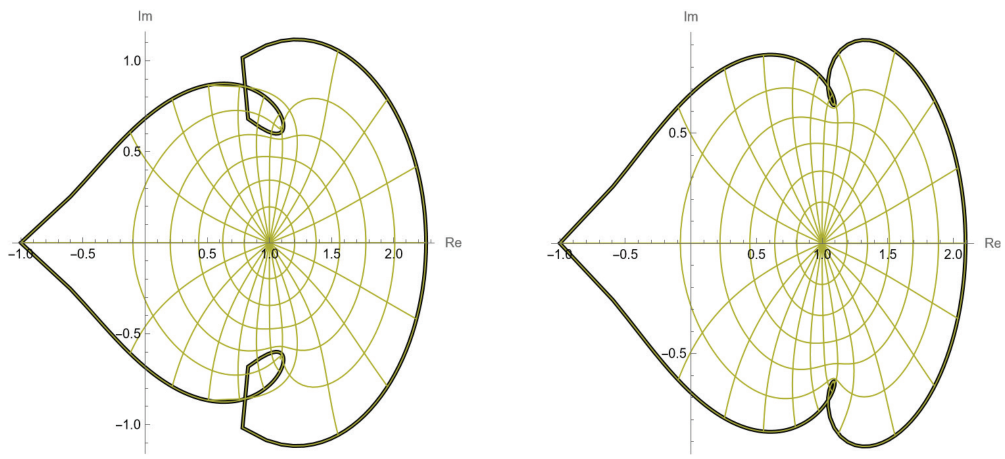
Inspired by the articles of Soboh and Oros [7] and Amourah et al. [14], we apply the discussed ideas and make use of the subordinating idea q-calculus. We thereby derive a bi-convex class connected with the leaf-like domain inside the open unit disc $\mathbb{U}$.

Definition 1.3: *A function $f\in\mathcal{S}$ is considered a member of the class $\mathcal{C}(q;\aleph(z;\,q))$ if it has the form (1) and satisfies the following subordinations:*

$$1+\frac{z\partial_q^2 f(z)}{\partial_q f(z)}\prec\aleph(z;\,q),\qquad (z\in\mathbb{U}), \tag{9}$$

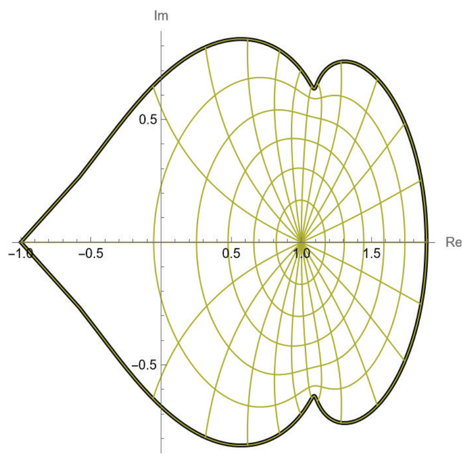
where $\aleph(z;q)$ of the form (7).

Remark 1.4: *The unit disk $\mathbb{U}$ undergoes a transformation into a region with a leaf-like shape using the analytic and univalent function $\aleph(z;q)$. This function exhibits symmetry with respect to the real axis and fulfills the conditions $\aleph(0;\,q)=\partial_q\aleph(0;\,q)=1$.*

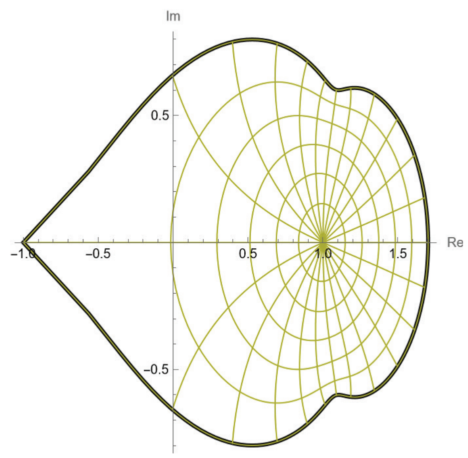


(a) The image of $\aleph(\mathbb{U};q)$ with $q\to 1^-$ (b) The image of $\aleph(\mathbb{U};0.89)$

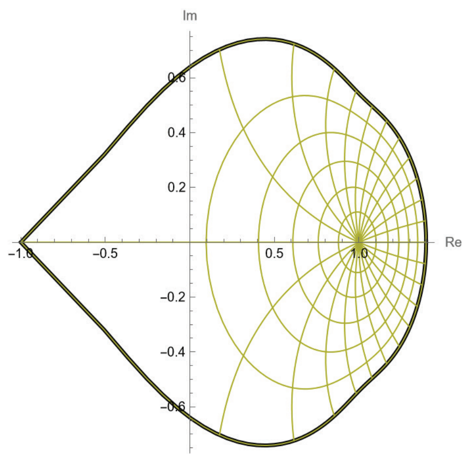
Figure 2: Continues.



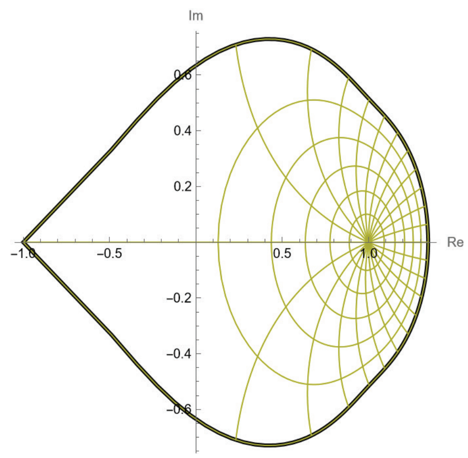
(c) The image of $\aleph(\mathbb{U};0.73)$



(d) The image of $\aleph(\mathbb{U};0.54)$



(e) The image of $\aleph(\mathbb{U};0.1)$



(f) The image of $\aleph(\mathbb{U})$ with $q \to 0^{+}$

Figure 2: The figure shows the leaf-shaped region $\aleph(\mathbb{U};\, q)$.

The primary objective of this work is to investigate the properties of bi-univalent functions associated with the leaf-like domain in $\mathbb{U}$. This section, along with supporting examples, provides a clear definition of the class under consideration. In Section 3, we derive coefficient estimates and analyze the Fekete-Szegö functional for the newly introduced class. Furthermore, leveraging the theorems established in the preceding sections, we present corollaries that align with the specific scenarios under discussion.

2. The Bi-Convex Class $\Sigma_{\kappa}\left(\aleph(z;q)\right)$

In this section, we employ the q-difference operator and the subordination principle between analytic functions, which were previously mentioned, to present a rigorous mathematical characterization of the newly introduced class $\Sigma_{\kappa}\left(\aleph(z;q)\right)$ of bi-univalent functions affiliated with a leaf-like domain.

Definition 2.1: *A function f of the type (1) belongs to the class $\Sigma_{\kappa}\left(\aleph(z;q)\right)$ if it satisfies the specified subordination conditions:*

$$\Phi(z;\, q) = 1 + \frac{z\partial_q^2 f(z)}{\partial_q f(z)} \prec \aleph(z;\, q), \quad (z \in \mathbb{U}), \tag{10}$$

and

$$\chi(\varsigma;q) = 1 + \frac{\varsigma \partial_q^2 g(\varsigma)}{\partial_q g(\varsigma)} \prec \aleph(\varsigma;q) \quad (\varsigma \in \mathbb{U}), \tag{11}$$

with $\aleph(z;q)$ *of the form (7) and* $g = f^{-1}$.

Remark 2.2: *We wish to underscore that the class* $\Sigma_{\kappa}\left(\aleph(z;q)\right)$ *is non-empty. To elaborate, let us examine the functions defined by:*

$$f_*(z) = \frac{z}{1-\vartheta z}, \quad |\vartheta| < 0.55. \tag{12}$$

It is evident that $f_* \in \mathcal{S}$ *and, furthermore,* $f_* \in \Sigma$ *together with its inverse:*

$$f_*^{-1}(\varsigma) = \frac{\varsigma}{1+\vartheta\varsigma}, \quad |\vartheta| < 0.55, \tag{13}$$

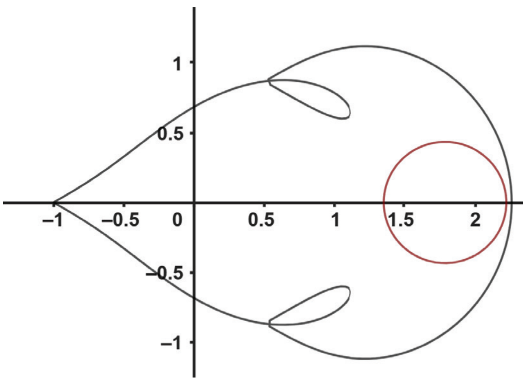
via means of the notations defined in equations (10) and (11), we may determine, via a basic computation, that:

$$\begin{aligned} \Phi(f_*(z);q) &= \frac{\vartheta}{1-q^2\vartheta z} + 1, \quad |\vartheta| < 0.55 \\ \chi(f_*^{-1}(\varsigma);q) &= \frac{\vartheta}{1+q^2\vartheta\varsigma} + 1, \quad |\vartheta| < 0.55. \end{aligned} \tag{14}$$

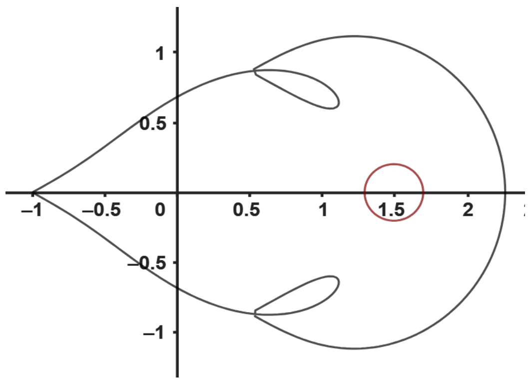
Also, for all $z \in \mathbb{U}$, $\Phi(-\vartheta z;q) = \chi(\vartheta z;q)$, *which implies that* $\Phi(\mathbb{U};q) = \chi(\mathbb{U};q)$.

We utilize GeoGebra Classic 6 to generate visual representations of the boundary denoted as $\partial\mathbb{U}$, employing the functions $\aleph$ and χ, as presented in Table 3. This approach is particularly applicable in situations where the conditions $|\vartheta| \le 0.55$ and $q \in (0,1)$ are satisfied. It is essential to acknowledge that $\aleph$ is a univalent function in $\mathbb{U}$. Consequently, the relationships $\Phi(z;q) \prec \aleph(z;q)$ and $\chi(z;q) \prec \aleph(z;q)$ are valid. These relationships are substantiated by the following facts: $\Phi(0;q) = \chi(0;q) = \aleph(0;q) = 1$, $\Phi(\mathbb{U};q) \subset \aleph(\mathbb{U};q)$, and $\chi(\mathbb{U};q) \subset \aleph(\mathbb{U};q)$. For further clarification, please refer to Figure 3.

Example 2.3: *if* $q \to 1^-$, *then* $\Sigma_{\kappa}(\aleph(z;q))$ *is reduced to* $\Sigma_{\kappa}\left(z + \sqrt[3]{1+z^3}\right)$ *defined by*

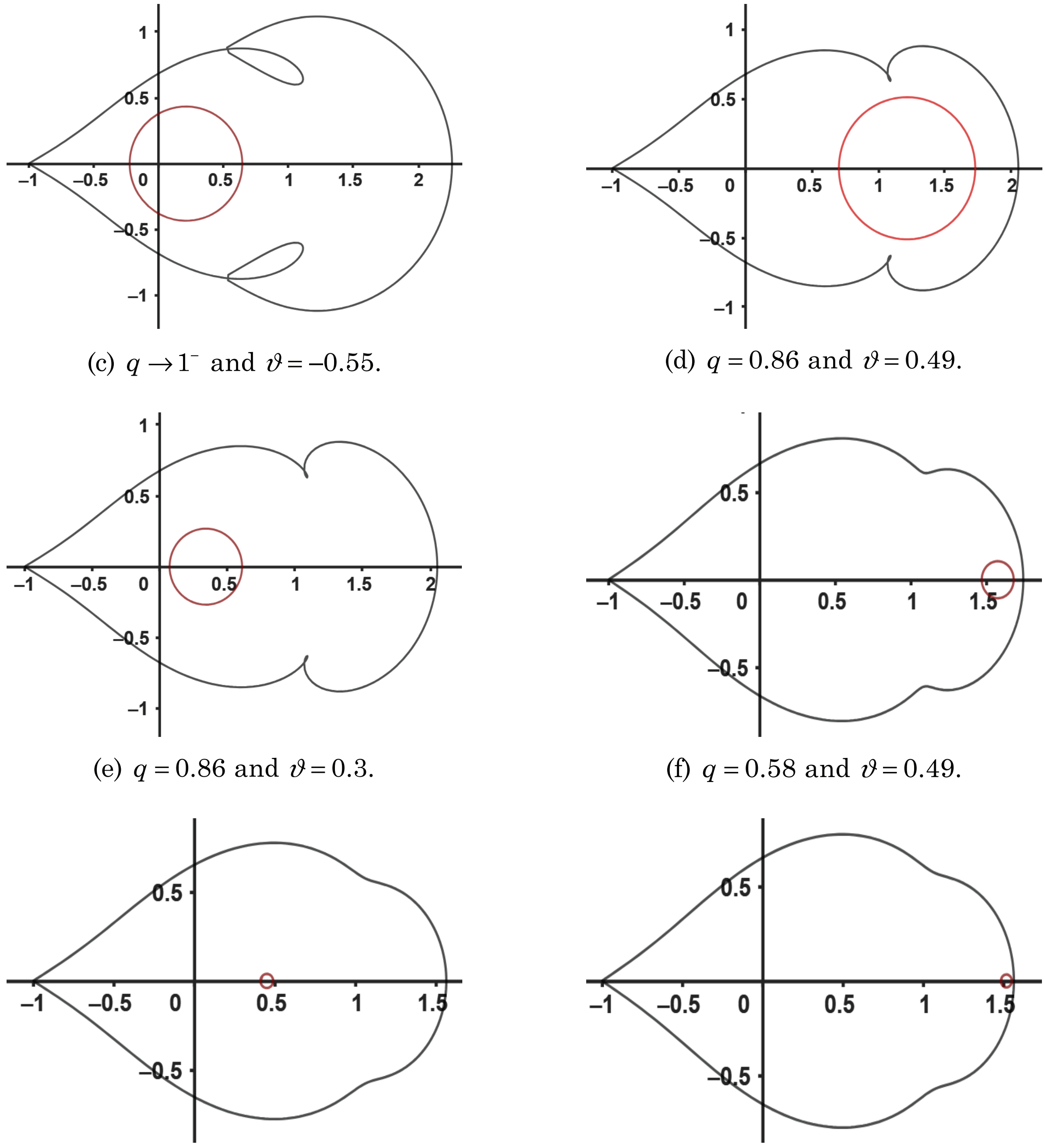


(a) $q \to 1^-$ and $\vartheta = 0.49$.



(b) $q \to 1^-$ and $\vartheta = 0.41$.

Figure 2: Continues.



(c) $q \to 1^-$ and $\vartheta = -0.55$.

(d) $q = 0.86$ and $\vartheta = 0.49$.

(e) $q = 0.86$ and $\vartheta = 0.3$.

(f) $q = 0.58$ and $\vartheta = 0.49$.

(g) $q = 0.36$ and $\vartheta = -0.55$.

(h) $q = 0.36$ and $\vartheta = 0.51$.

Figure 3: The image of $\aleph(e^{i\theta};q)$ (black color) and $\Phi(e^{i\theta};q)$ (red color) with several values of q, ϑ and $\theta \in [0, 2\pi)$.

$$\left\{ f \in \Sigma : \frac{z f''(z)}{f'(z)} + 1 \prec z + \sqrt[3]{1+z^3}, \quad (z \in \mathbb{U}) \right\}, \tag{15}$$

and

$$\left\{ f \in \Sigma : \frac{\varsigma\, (f^{-1})''(\varsigma)}{f'(\varsigma)} + 1 \prec \varsigma + \sqrt[3]{1+\varsigma^3}, \quad (\varsigma \in \mathbb{U}) \right\}. \tag{16}$$

3. The bounds of the coefficients and the Fekete-Szegö functional within the class $\Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$

Theorem 3.1: *Let $f \in \Sigma$ of the form (1) be included in the class $\Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$. Then*

$$|a_2| \leq \frac{1+q}{\sqrt{8[2]_q\left[[3]_q(1+q)-4[2]_q\right]}},$$

and

$$|a_3| \leq \frac{1+q}{4[2]_q[3]_q} + \frac{(1+q)^2}{16\left([2]_q\right)^2}.$$

proof: If $f \in \Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$, then, as per Definition 2.1, the existence of certain analytic functions $\varkappa_1$ and $\varkappa_2$ may be established. These functions fulfil the requirements $\varkappa_1(0) = \varkappa_2(0) = 0$, and $|\varkappa_1(z)| < 1$, $|\varkappa_2(\varsigma)| < 1$ for every $z, \varsigma \in \mathbb{U}$.

$$\delta_1(z) = \frac{1+\varkappa_1(z)}{1-\varkappa_1(z)} = 1 + \ell_1 z + \ell_2 z^2 + \cdots, \ (z \in \mathbb{U}),$$

and

$$\delta_2(\varsigma) = \frac{1+\varkappa_2(\varsigma)}{1-\varkappa_2(\varsigma)} = 1 + \jmath_1 \varsigma + \jmath_2 \varsigma^2 + \cdots, \ (\varsigma \in \mathbb{U}),$$

then $\delta_1, \delta_2 \in P$. From the above relations, we get

$$\varkappa_1(z) = \frac{\delta_1(z)-1}{\delta_1(z)+1}, \quad (z \in \mathbb{U}),$$

and

$$\varkappa_2(\varsigma) = \frac{\delta_2(\varsigma)-1}{\delta_2(\varsigma)+1}, \quad (\varsigma \in \mathbb{U}).$$

From (10) and (11) it follows that

$$\begin{aligned}\aleph(\varkappa_1(z);\, q) &= \frac{(1+q)\left(\delta_1(z)-1\right)}{1+3\delta_1(z)+q\left(1-\delta_1(z)\right)} + \sqrt[3]{1+\left(\frac{(1+q)\left(\delta_1(z)-1\right)}{1+3\delta_1(z)+q\left(1-\delta_1(z)\right)}\right)^3} \\ &= 1 + \frac{1+q}{4}\ell_1 w + \frac{1+q}{4}\left(\ell_2 - \frac{(3-q)}{4}\ell_1^2\right)z^2 + \cdots, \qquad \left(z \in \mathbb{U}\right),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\aleph(\varkappa_2(\varsigma);q) &= \frac{(1+q)\left(\delta_2(\varsigma)-1\right)}{1+3\delta_2(\varsigma)+q\left(1-\delta_2(\varsigma)\right)} + \sqrt[3]{1+\left(\frac{(1+q)\left(\delta_2(\varsigma)-1\right)}{1+3\delta_2(\varsigma)+q\left(1-\delta_2(\varsigma)\right)}\right)^3} \\ &= 1 + \frac{1+q}{4}\jmath_1 \varsigma + \frac{1+q}{4}\left(\jmath_2 - \frac{(3-q)}{4}\jmath_1^2\right)\varsigma^2 + \cdots, \qquad \left(\varsigma \in \mathbb{U}\right).\end{aligned} \tag{18}$$

Also,

$$1 + \frac{z\partial_q^2 f(z)}{\partial_q f(z)} = 1 + \frac{1+q}{4}\ell_1 z + \frac{1+q}{4}\left(\ell_2 - \frac{(3-q)}{4}\ell_1^2\right)z^2 + \cdots, \tag{19}$$

and

$$1+\frac{\varsigma\partial_q^2 g(\varsigma)}{\partial_q g(\varsigma)}=1+\frac{1+q}{4}\jmath_1\varsigma+\frac{1+q}{4}\left(\jmath_2-\frac{(3-q)}{4}\jmath_1^2\right)\varsigma^2+\cdots, \tag{20}$$

By comparing the relevant coefficients in equation (19) and equation (20), we can deduce the following.

$$[2]_q a_2=\frac{1+q}{4}\ell_1, \tag{21}$$

$$-[2]_q a_2=\frac{1+q}{4}\jmath_1, \tag{22}$$

$$[2]_q[3]_q a_3-[2]_q^2 a_2^2=\frac{1+q}{4}\left(\ell_2-\frac{(3-q)}{4}\ell_1^2\right), \tag{23}$$

and

$$[2]_q\left(2[3]_q-[2]_q\right)a_2^2-[2]_q[3]_q a_3=\frac{1+q}{4}\left(\jmath_2-\frac{(3-q)}{4}\%\jmath_1^2\right). \tag{24}$$

It follows from (21) and (22) that

$$\ell_1=-\jmath_1 \quad \text{and} \quad \ell_1^2=\jmath_1^2, \tag{25}$$

and

$$2\left([2]_q\right)^2 a_2^2=\frac{(1+q)^2}{16}\left(\ell_1^2+\jmath_1^2\right)a_2^2=\frac{(1+q)^2}{32\left([2]_q\right)^2}\left(\ell_1^2+\jmath_1^2\right) \iff \ell_1^2+\jmath_1^2=\frac{32\left([2]_q\right)^2}{(1+q)^2\%}a_2^2.$$

Adding (23) and (24), with doing some calculations, we get

$$2[2]_q\left([3]_q-[2]_q\right)a_2^2=\frac{1+q}{4}\left[(\ell_2+\jmath_2)+\frac{(3-q)}{4}\left(\ell_1^2+\jmath_1^2\right)\right].$$

Substituting the value of $\left(\ell_1^2+\jmath_1^2\right)$ from (26), we obtain

$$2[2]_q\left[[3]_q-\left(1+\frac{(3-q)}{(1+q)}\right)[2]_q\right]a_2^2=\frac{1+q}{8}(\ell_2+\jmath_2).$$

Moreover,

$$a_2^2=\frac{1+q}{16[2]_q\left[[3]_q-\left(1+\frac{(3-q\%)}{(1+q)}\right)[2]_q\right]}(\ell_2+\jmath_2). \tag{27}$$

Applying (4) for the coefficients ℓ_2 and $\jmath_2$, we obtain

$$|a_2|\le\frac{1+q}{\sqrt{8[2]_q\left[[3]_q(1+q)-4[2]_q\right]}},$$

By subtracting (24) from (23), and using $\ell_1^2=\jmath$, we get

$$2[2]_q[3]_q\left(a_3-a_2^2\right)=\frac{1+q}{4}\left(\ell_2-\jmath_2\right). \tag{28}$$

Then, in view of (26), the (28) becomes

$$a_3 = \frac{1+q}{8[2]_q[3]_q}\left(\ell_2 - \jmath_2\right) + \frac{(1+q)^2}{32\left([2]_q\right)^2}\left(\ell_1^2 + \jmath_1^2\right).$$

Thus applying (4), we conclude that

$$|a_3| \le \frac{1+q}{4[2]_q[3]_q} + \frac{(1+q)^2}{16\left([2]_q\right)^2}.$$

□

Theorem 3.2: *For any $\mu \in \mathbb{R}$, consider f defined as in equation (1), which belongs to the class $\Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$. Then,*

$$[\left|a_3 - \mu a_2^2\right| \le \%\begin{cases} \dfrac{(1+q)}{4[2]_q[3]_q}, & |1-\mu| \le 2\left|1 - \dfrac{4[2]_q}{[3]_q(1+q)}\right|, \\[2ex] \dfrac{(1+q)}{4[2]_q}\left|\mathcal{Y}(\mu)\right|, & |1-\mu| \ge 2\left|1 - \dfrac{4[2]_q}{[3]_q(1+q\%)}\right|. \end{cases}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)}{2\left[[3]_q(1+q) - 4[2]_q\right]}.$$

proof: If $f \in \mathcal{K}(\aleph(z;\, q))$ is expressed in the manner (1), then from (27) and (28), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\mu)(1+q)^2}{16[2]_q\left[[3]_q(1+q) - 4[2]_q\right]}\left(\ell_2 + \jmath_2\right) + \frac{1+q}{8[2]_q[3]_q}\left(\ell_2 - \jmath_2\right) \\ &= \frac{(1+q)}{8[2]_q}\left[\left(\mathcal{Y}\%(\mu) + \frac{1}{[3]_q}\right)\ell_2 + \left(\mathcal{Y}(\mu) - \frac{1}{[3]_q}\right)\jmath_2\right] \end{aligned}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)}{2\left[[3]_q(1+q) - 4[2]_q\right]}.$$

Then, we conclude that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{(1+q)}{4[2]_q[3]_q}, & \left|\mathcal{Y}(\mu)\right| \le \dfrac{1}{[3]_q}, \\[2ex] \dfrac{(1+q)}{4[2]_q}\left|\mathcal{Y}(\mu)\right|, & \left|\mathcal{Y}(\mu)\right| \ge \dfrac{1}{[3]_q}. \end{cases}$$

This finishes the proof of Theorem 3.2. □

Theorems 3.1 and 3.2 yield the following corollary, which is typically associated with Example 2.3.

Corollary 3.3: *For any $\mu \in \mathbb{R}$, consider f defined as in equation (1), which belongs to the class $\Sigma_{\kappa}(\aleph(z;q))$. Then, we can state the following*

$$|a_2| \leq \frac{1}{2\sqrt{2}}, \qquad |\varrho_3| \leq \frac{7}{48}, \quad and \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{12}, & |1-\mu| \leq \frac{2}{3}, \\ \frac{1}{8}|1-\mu|, & |1-\mu| \geq \frac{2}{3}. \end{cases}$$

4. Conclusion

This paper investigates coefficient issues pertaining to a newly established subclass of bi-univalent functions within the domain $\mathbb{U}$, as specified in Definition 2.1. This subclass, represented by $K(\aleph(z;\, q))$, has been extensively analysed. We have established bounds for the second and third Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, for functions inside this class. Furthermore, we have supplied estimations for the Fekete-Szegö functional, therefore augmenting our comprehension of the geometric and analytic characteristics of these functions. These findings establish a foundation for additional research into coefficient inequalities and associated extremal issues within the theory of bi-univalent functions.

In future studies, one could investigate the maximum bounds of the Zalcman conjecture and analyze Hankel determinants for the classes of bi-convex and bi-close-to-convex functions. These directions present promising opportunities for novel discoveries and deeper exploration in the field.

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