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Some novel versions of fractional hermite–hadamard-mercer type inequalities with matrix applications

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Abstract

In this study, we explore several fractional Hermite–Hadamard (H–H)-Mercer inequalities for interval-valued functions through the use of a generalized fractional integral operator (GFIO). Furthermore, we examine new variations of the H–H-Mercer inequality in relation to GFIO. Various examples are included to support our assertions. The results could offer new insights into a broad spectrum of integral inequalities for fractional fuzzy systems within the framework of interval analysis, along with the optimization issues they raise. Moreover, some applications on matrices are illustrated.

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1. Introduction

The study of convexity in the context of fractional calculus has been a significant area of research for over a century, attracting considerable attention across mathematics and various scientific disciplines. Its geometric interpretation provides valuable insights and practical results that extend to numerous fields. Additionally, convexity offers powerful tools and numerical methods that help in

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solving a wide array of interconnected problems. In applied mathematics, fractional analysis, particularly involving convexity, has found extensive and noteworthy applications.

Convex inequalities refer to mathematical inequalities that involve convex functions. Convexity is a widely recognized concept that plays a crucial role in various domains, including economics, finance, game theory, optimization, quality management, statistical theory and many other sciences. Given its broad applicability, the concept of convexity has been generalized in multiple ways. This theory has been at the heart of significant mathematical research for over a century. The intersection of convexity and optimization has made a profound impact on numerous applied sciences, containing control systems [1], mathematical optimization in modeling (see [2, 3]), estimation and signal processing [4], finance [5], as well as data analysis and computer science. [6].

Inequalities are the essential tools due to their importance in fractional calculus, traditional calculus, quantum calculus, stochastic, time-scale calculus, fractal sets, and other fields. The crucial mathematical tools connecting inequalities and integrals are integral inequalities. These inequalities give valuable understanding of how functions behave within specific intervals. They serve as a versatile approach to grasping growth trends, convergence properties, and approximations of functions.

These mathematical tools find widespread use in various disciplines, including economics, physics, engineering, probability theory and information technology. For specific applications, refer to ([7, 20]). Integral inequalities facilitate the estimation of values that may be difficult to calculate directly by employing bounding functions. They also play a key role in proving the convergence of series and sequences, the existence of optimal solutions in optimization tasks and the stability of solutions of differential equations. By bridging integrals and inequalities, these tools offer a powerful and elegant framework for tackling and analyzing complex problems.

A fractional H-H inequality was introduced in [21]. The study of famous inequalities, like Simpson, Ostrowski and Hadamard laid the groundwork for the development of fractional integral inequalities. Fractional calculus finds applications in various fields, including modelling, engineering, transform theory, mathematical biology, finance, image processing, natural phenomenon prediction, healthcare and fluid dynamics. For additional details, (see [22–25]). Fractional integral inequalities are mathematical expressions that expand classical integral operators to non-integer orders by establishing boundaries or relationships between functions utilizing fractional integrals. The theory of fractional calculus, which deals with integrals and derivatives of arbitrary order, heavily relies on these inequalities. Fractional integral inequalities offer strong instruments for examining the behavior of solutions to fractional differential equations by expanding on classical results like Holder, Minkowski, and Grönwall inequalities. In mathematical physics, control theory, and engineering models where memory and hereditary qualities are crucial, they are especially significant when studying the stability, uniqueness, and existence of solutions.

In second section, we review the essential definitions, remarks and theorems that are needed for the subsequent sections. Section 3 provides some notations related to interval analysis, along with the necessary background data. Section 4 is devoted to explore new variants of the H-H-Mercer type inequality for convex interval-valued functions (CIVFs) using the generalized fractional integral operator (GFIO), and we present new corollaries. We also discuss remarks demonstrating that our results are more general and novel. In Section 5, we provide applications using matrix formulations to illustrate the newly developed results. The last section presents a conclusion.

2. Preliminaries

At first, we recall and examine few definitions, remarks, and theorems, which will be essential for the paper.

Jensen [26] introduced the concept of convexity.

Definition 2.1 [26, 27] *A real-valued function* $\mathfrak{f}:[\mathfrak{a},\mathfrak{b}]\to\mathbb{R}$ *is named as convex, if for all* $\Lambda,\mathcal{Y}\in[\mathfrak{a},\mathfrak{b}]$ *and* $\wp\in[0,1]$,

$$\mathfrak{f}\big(\wp\Lambda+(1-\wp)\mathcal{Y}\big)\le\wp\mathfrak{f}(\Lambda)+(1-\wp)\mathfrak{f}(\mathcal{Y}).$$

Let $\mathcal{I}$ be an interval in $\mathbb{R}$. We give the H-H inequality [28]: If the function $\mathfrak{f}:\mathcal{I}\to\mathbb{R}$ is convex, then for $\Lambda,\mathcal{Y}\in\mathcal{I}$ with $\mathcal{Y}>\Lambda$, we have

$$\mathfrak{f}\Big(\frac{\Lambda+\mathcal{Y}}{2}\Big)\le\frac{1}{\mathcal{Y}-\Lambda}\int_{\Lambda}^{\mathcal{Y}}\mathfrak{f}(\wp)d\wp\le\frac{\mathfrak{f}(\Lambda)+\mathfrak{f}(\mathcal{Y})}{2}. \tag{1}$$

This inequality provides an upper bound for the value of the function at the midpoint of Λ and $\mathcal{Y}$ in terms of the values of the function at the endpoints of the interval. This inequality, which can be proved under fairly simple conditions, is commonly applied by researchers in fields like information theory and inequality theory. We state the Jensen inequality.

Consider the reals $0<\gimel_1\le\gimel_2\le\ldots\le\gimel_n$. Choose $\varpi=(\varpi_1,\varpi_2,\cdots,\varpi_n)$ as non-negative weights so that $\sum_{q=1}^{n}\varpi_q=1$. Following [29]), $\mathfrak{f}$ is termed as convex on $[\Lambda,\mathcal{Y}]$, when

$$\mathfrak{f}\left(\sum_{q=1}^{n}\varpi_q\,\gimel_q\right)\le\left(\sum_{q=1}^{n}\varpi_q\,\mathfrak{f}(\gimel_q)\right),$$

for all $\varpi_q\in[0,1]$, $\gimel_q\in[\Lambda,\mathcal{Y}]$ with $(q=1,2,\ldots,n)$.

This inequality has numerous applications in information theory (see [30]). However, while many researchers focused on the Jensen inequality, the modification introduced by Mercer stands out as particularly significant and unique. In 2003, Mercer [31] explored a new version of Jensen inequality.

If $\mathfrak{f}$ is convex on $[\Lambda,\mathcal{Y}]$, then

$$\mathfrak{f}\left(\Lambda+\mathcal{Y}-\sum_{q=1}^{n}\varpi_q\gimel_q\right)\le\mathfrak{f}(\Lambda)+\mathfrak{f}(\mathcal{Y})-\sum_{q=1}^{n}\varpi_q\,\mathfrak{f}(\gimel_q),$$

for all $\varpi_q\in[0,1]$, $\gimel_q\in[\Lambda,\mathcal{Y}]$ with $(q=1,\ldots,n)$.

Pečarić et al. [32] developed numerous modifications to the fascinating topic of Jensen-Mercer operator inequalities. Following their work, Niezgoda [33] introduced new variants of inequalities of Mercer-type with higher dimensions. Due to its significant properties, the inequality of Jensen-Mercer type has a notable contribution to the theory of inequalities. Kian [34] examined and explored the Jensen inequality in relation to superquadratic functions.

In [35], the authors presented the next Hermite-Hadamard-Mercer (H-H-Mercer) inequality:

$$\begin{aligned}\mathfrak{f}\left(\Lambda+\mathcal{Y}-\frac{u_1+u_2}{2}\right)&\le\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\mathfrak{f}(\Lambda+\mathcal{Y}-\wp)d\wp\\&\le\frac{\mathfrak{f}(\Lambda+\mathcal{Y}-u_1)+\mathfrak{f}(\Lambda+\mathcal{Y}-u_2)}{2}\le\mathfrak{f}(\Lambda)+\mathfrak{f}(\mathcal{Y})-\frac{\mathfrak{f}(u_1)+\mathfrak{f}(u_2)}{2}.\end{aligned}$$

Here, the function $\mathfrak{f}$ is convex on $[\mathfrak{a},\mathfrak{b}]$. For more literature regarding the above inequality, see [36–38].

There are two types of non-local fractional derivatives: the Caputo and Riemann-Liouville derivatives having singular kernels, and the Atangana-Baleanu and Caputo-Fabrizio derivatives having non-singular kernels. Operators with Fractional derivative having kernels that are non-singular are useful in addressing non-locality in practical domains. Let $\mathcal{L}[\mathfrak{a},\mathfrak{b}]$ denote the set of Leabesgue integrable functions on $[\mathfrak{a},\mathfrak{b}]$.

Definition 2.2 [39] *Let $\mathfrak{f}\in\mathcal{L}[\mathfrak{a},\mathfrak{b}]$. We define the left-sided and right-sided Riemann-Liouville fractional integrals of order $\ell>0$ as*

$${}_{\mathfrak{a}}\mathfrak{J}^{\ell}\mathfrak{f}(\tau)=\frac{1}{\Gamma(\ell)}\int_{\mathfrak{a}}^{\tau}(\tau-\mu)^{\ell-1}\mathfrak{f}(\mu)d\mu,\qquad\mathfrak{a}<\tau \tag{2}$$

and

$$\mathfrak{J}_{\mathfrak{b}}^{\wp}\mathfrak{f}(\tau)=\frac{1}{\Gamma(\ell)}\int_{\tau}^{\mathfrak{b}}(\mu-\tau)^{\ell-1}\mathfrak{f}(\mu)d\mu, \qquad \tau<\mathfrak{b}, \tag{3}$$

where $\Gamma(\ell)=\int_0^{\infty}e^{-u}u^{\ell-1}du$ is the gamma function.

The next generalized fractional integral operator is investigated by Jarad et al. [40].

$$ {}^{\beta}_{\mathfrak{a}}\mathfrak{J}^{\ell}\mathfrak{f}(x)=\frac{1}{\Gamma(\beta)}\int_{\mathfrak{a}}^{x}\left(\frac{(x-\mathfrak{a})^{\ell}-(\wp-\mathfrak{a})^{\ell}}{\ell}\right)^{\beta-1}\frac{\mathfrak{f}(\wp)}{(\wp-\mathfrak{a})^{1-\ell}}d\wp \tag{4}$$

and

$$ {}^{\beta}\mathfrak{J}_{\mathfrak{b}}^{\ell}\mathfrak{f}(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\mathfrak{b}}\left(\frac{(\mathfrak{b}-x)^{\ell}-(\mathfrak{b}-\wp)^{\ell}}{\ell}\right)^{\beta-1}\frac{\mathfrak{f}(\wp)}{(\mathfrak{b}-\wp)^{1-\ell}}d\wp, \tag{5}$$

where $\mathfrak{b}>\mathfrak{a}$ and $\wp\in[0,1]$.

Recently, numerous integral inequalities have been studied and analyzed using interval-valued functions (IVFs). For more information, refer to the literature [41–46].

3. Operations on intervals

A real-valued interval is denoted by $Z=[\mu,\nu]=\{x\in\mathbb{R} \mid \mu\le x\le\nu\}$, where $\mu,\nu\in\mathbb{R}$ with $\mu\le\nu$. Here, μ and ν are termed as left and right endpoints of Z, respectively. If $a=b$, the interval X is called degenerate. Here, the notation $Z=e=[e,e]$ is used. Also, an interval Z is called positive if $\mu>0$ and Z is called negative if $\nu<0$. The family of all closed (resp. closed positive, closed negative) intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_I$ (resp. $\mathbb{R}_I^+$, $\mathbb{R}_I^-$). The Hausdorff-Pompieu distance between $\mathbf{U}=[\mu,\nu]$ and $\mathbf{W}=[\tau,\sigma]$ is known as $d_H(\mathbf{U},\mathbf{W})=\max\{|\mu-\tau|,|\nu-\sigma|\}$.

Let $\mathbf{U}=[\mu,\nu]$ and $\mathbf{W}=[\tau,\sigma]$. We have the next operations (see [46]):

- **Addition:**

$$\mathbf{U}+\mathbf{W}=[\mu+\tau,\nu+\sigma].$$

- **Subtraction:**

$$\mathbf{U}-\mathbf{W}=[\mu-\sigma,\nu-\tau].$$

- **Multiplication:**

$$\mathbf{U}\cdot\mathbf{W}=[\min(\mu\tau,\mu\sigma,\nu\tau,\nu\sigma),\max(\mu\tau,\mu\sigma,\nu\tau,\nu\sigma)].$$

- **Division (if $0\notin\mathbf{W}$):**

$$\frac{\mathbf{U}}{\mathbf{W}}=[\min(\frac{\mu}{\tau},\frac{\mu}{\sigma},\frac{\nu}{\tau},\frac{\nu}{\sigma}),\max(\frac{\mu}{\tau},\frac{\mu}{\sigma},\frac{\nu}{\tau},\frac{\nu}{\sigma})].$$

3.1. Remarks

We also have the next algebraic properties (see [50, 56]).

1. The set of intervals is closed under subtraction, addition, multiplication, and division (if $0 \notin \mathbf{W}$):

$$\mathbf{U}+\mathbf{W},\quad \mathbf{U}-\mathbf{W},\quad \mathbf{U}\cdot\mathbf{W},\quad \frac{\mathbf{U}}{\mathbf{W}} \quad \text{areallintervals.}$$

2. Commutativity:

$$\mathbf{U}+\mathbf{W}=\mathbf{W}+\mathbf{U},$$
$$\mathbf{U}\cdot\mathbf{W}=\mathbf{W}\cdot\mathbf{U}.$$

3. Associativity:

$$(\mathbf{U}+\mathbf{W})+\mathbf{T}=\mathbf{U}+(\mathbf{W}+\mathbf{T}),$$
$$(\mathbf{U}\cdot\mathbf{W})\cdot\mathbf{T}=\mathbf{U}\cdot(\mathbf{W}\cdot\mathbf{T}).$$

4. Distributivity of multiplication over addition:

$$\mathbf{U}\cdot(\mathbf{W}+\mathbf{T})=(\mathbf{U}\cdot\mathbf{W})+(\mathbf{U}\cdot\mathbf{T}).$$

5. $[0,0]$ is the additive identity :

$$\mathbf{U}+[0,0]=\mathbf{U}.$$

6. $[1,1]$ is the multiplicative identity :

$$\mathbf{U}\cdot[1,1]=\mathbf{U}.$$

7. The additive inverse of $\mathbf{U}=[\mu,\nu]$ is $-\mathbf{U}=[-\mu,-\nu]$, that is,

$$\mathbf{U}+(-\mathbf{U})=[0,0].$$

8. Associative law: for all $\lambda,\omega\in\mathbb{R}$, $\lambda(\omega\mathbf{U})=(\lambda\omega)\mathbf{U}$.
9. First distribution law:

$$\lambda(\mathbf{U}+\mathbf{W})=\lambda\mathbf{U}+\lambda\mathbf{W}.$$

10. Second distribution law:

$$(\lambda+\omega)\mathbf{U}=\lambda\mathbf{U}+\omega\mathbf{U},\ \text{for all}\,\mathbf{U}\in\mathbb{R}_I\ \text{ and}\,\lambda\omega\geq 0.$$

11. Cancellation law:

$$\mathbf{U}\subseteq\mathbf{W}\mathbf{U}+\mathbf{T}\subseteq\mathbf{W}+\mathbf{T},$$

and

$$\mathbf{U}\subseteq\mathbf{W}\mathbf{U}\cdot\mathbf{T}\subseteq\mathbf{W}\cdot\mathbf{T}.$$

12. Interval multiplication is subdistributive over interval addition:

$$\mathbf{U}\cdot(\mathbf{W}+\mathbf{T})\subseteq(\mathbf{U}\cdot\mathbf{W})+(\mathbf{U}\cdot\mathbf{T}).$$

In generally, the distributive law is not always applied to intervals. For instance, consider

$$\mathbf{U}=[1,2],\ \ \mathbf{W}=[2,3],\ \ \mathbf{T}=[-2,-1].$$

We have $\mathbf{U}.(\mathbf{W}+\mathbf{T})=[0,4]$, while $\mathbf{U}.\mathbf{W}+\mathbf{U}.\mathbf{T}=[-2,5]$.

3.2. Integral of interval-valued functions

This part introduces the concept of integrals for interval-valued functions (IVFs) (see [17]). First, a mapping $\mathfrak{f}$ is called an IVF of $[\Lambda,\Omega]$ if it is assigned a nonempty interval to every $\mathcal{K}\in[\Lambda,\Omega]$, that is,

$$\mathfrak{f} = [\underline{\mathfrak{f}}(\mathcal{K}), \overline{\mathfrak{f}}(\mathcal{K})], \ \textit{for each } \mathcal{K} \in [\Lambda, \Omega],$$

where $\overline{\mathfrak{f}}$ and $\underline{\mathfrak{f}}$ are real-valued mappings. A partition of $[\Lambda, \Omega]$ is every ordered finite subset $\mathcal{P}$ with the form

$$\mathcal{P} : \Lambda = \mathcal{K}_0 < \mathcal{K}_1 < \cdots < \mathcal{K}_d = \Omega.$$

The mesh (or norm) of a partition $\mathcal{P}$ is given as

$$mesh(\mathcal{P}) = \max\{\mathcal{K}_i - \mathcal{K}_{i-1} : i = 1, 2, ..., d\} = ||\mathcal{P}||.$$

Here, the notation $\mathcal{P}([\Lambda, \Omega])$ corresponds to the family of all partitions of $[\Lambda, \Omega]$. Denote by $\mathcal{P}(\delta, [\Lambda, \Omega])$ the set of elements P in $\mathcal{P}([\Lambda, \Omega])$ so that $mesh(P) < \delta$.

Let β_i be arbitrary in each $[\mathcal{K}_{i-1}, \mathcal{K}_i]$ $(i = 1, \cdots, d)$ and given $\mathfrak{f} : [\Lambda, \Omega] \to \mathbb{R}_I$. Consider

$$\mathcal{S}(\mathfrak{f}, P, \delta) = \sum_{i=1}^{d} \mathfrak{f}(\delta_i)(\mathcal{K}_i - \mathcal{K}_{i-1}).$$

$\mathcal{S}(\mathfrak{f}, P, \delta)$ is termed as a Riemann sum of $\mathfrak{f}$ associated with $P \in \mathcal{P}([\Lambda, \Omega])$.

Definition 3.1 *[49] The mapping $\mathfrak{f} : [\Lambda, \Omega] \to \mathbb{R}_I$ is named as an interval Riemann integrable (IR-integrable) on $[\Lambda, \Omega]$, if there is $\mathcal{A} \in \mathbb{R}_I$ and for every $\varepsilon > 0$, there exists $\delta > 0$ so that*

$$d(\mathcal{S}(\mathfrak{f}, P, \delta), \mathcal{A}) < \varepsilon$$

for any $P \in \mathcal{P}(\delta, [\Lambda, \Omega])$. Here, $\mathcal{A}$ is named an $IR-$integral of $\mathfrak{f}$ on $[\Lambda, \Omega]$. Consider

$$\mathcal{A} = (IR)\int_{\Lambda}^{\Omega} \mathfrak{f}(\wp)d\wp. \tag{6}$$

Theorem 3.2 *[56] Given an IVF $\mathfrak{f} : [\Lambda, \Omega] \to \mathbb{R}_I$ with $\mathfrak{f}(\wp) = [\underline{\mathfrak{f}}(\wp), \overline{\mathfrak{f}}(\wp)]$. The mapping $\mathfrak{f} \in IR_{([\Lambda,\Omega]}$ iff $\underline{\mathfrak{f}}(\wp), \overline{\mathfrak{f}}(\wp) \in IR_{([\Lambda,\Omega]}$ and*

$$(IR)\int_{\Lambda}^{\Omega} \mathfrak{f}(\wp)d\wp = [(R)\int_{\Lambda}^{\Omega} \underline{\mathfrak{f}}(\wp)d\wp, (R)\int_{\Lambda}^{\Omega} \overline{\mathfrak{f}}(\wp)d\wp].$$

Definition 3.3 *[58] An IVF $\mathfrak{f} : [\Lambda, \Omega] \to \mathbb{R}_I$ is called a CIVF if*

$$\nu\mathfrak{f}(x) + (1-\nu)\mathfrak{f}(\upsilon) \subseteq \mathfrak{f}(\nu x + (1-\nu)\upsilon)$$

for all $x, \upsilon \in [\Lambda, \Omega]$ and $\nu \in (0,1)$.

Definition 3.4 [57] *Let $h : [c, d] \to \mathbb{R}$ be so that $h > 0$ and $(0,1) \subseteq [c, d]$. Then $\mathfrak{f} : [\Lambda, \Omega] \to \mathbb{R}_I^+$ is called h -CIVF, if*

$$h(\tau)\mathfrak{f}(\wp_1) + h(1-\tau)\mathfrak{f}(\wp_2) \subseteq \mathfrak{f}(\tau\wp_1 + (1-\tau)\wp_2), \tag{7}$$

for all $\wp_1, \wp_2 \in [\Lambda, \Omega]$ and $\tau \in (0,1)$.

Remark 3.5 *Taking $h(\tau) = \tau$ in Eq. (7), then we get the definition of a convex interval-valued function (CIVF) [58]. While, when $h(\tau) = \tau^s$ in Eq. (7), then we get an s-CIVF (see [59]).*

Zhao et al. [60] applied h-CIVFs within the framework of interval analysis. They also introduced the next H-H inequality. Mention that $SX(h-CIVF, [\Lambda, \Omega], \mathbb{R}_I^+)$ is the set of all $h-$CIVFs.

Theorem 3.6 [60] *If* $SX(h,[\Lambda,\Omega],\mathbb{R}_I^+)$ *and* $h(\frac{1}{2})\neq 0$, *then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right)\supseteq\frac{1}{\Omega-\Lambda}(IR)\int_\Lambda^\Omega \mathfrak{f}(\wp)d\wp\supseteq\left[\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\right]\int_0^1 h(\wp)d\wp. \tag{8}$$

Remark 3.7 *Choosing* $h(\vartheta)=\vartheta$ *in Eq. (8), one finds*

$$\mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right)\supseteq\frac{1}{\Omega-\Lambda}(IR)\int_\Lambda^\Omega \mathfrak{f}(x)dx\supseteq\frac{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)}{2}. \tag{9}$$

Eq. (9) was firstly introduced by Sadowska in [58].
Taking $h(\vartheta)=\vartheta^s$ in Eq. (8), one gets

$$2^{s-1}\mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right)\supseteq\frac{1}{\Omega-\Lambda}(IR)\int_\Lambda^\Omega \mathfrak{f}(x)dx\supseteq\frac{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)}{s+1}. \tag{10}$$

Osuna-Gómez et al. in [61] studied Eq.(10).

Theorem 3.8 [58] *An IVF* $\mathfrak{f}:[\Lambda,\Omega]\to\mathbb{R}_I$ *is called a CIVF, if and only if* $\underline{\mathfrak{f}}$ *(resp.* $\overline{\mathfrak{f}}$*) is convex (resp. concave) function on* $[\Lambda,\Omega]$.

Theorem 3.9 [62] *Let* $0<\beth_1\le\beth_2\le\cdots\le\beth_n$ *be reals and let* $\mathfrak{f}$ *be a CIVF on an interval including* ϖ_k. *Then*

$$\mathfrak{f}\left(\sum_{j=1}^n\varpi_j\,\beth_j\right)\supseteq\left(\sum_{j=1}^n\varpi_j\,\mathfrak{f}(\beth_j)\right), \tag{11}$$

where $\sum_{j=1}^n\varpi_j=1$, $\varpi_j\in[0,1]$.
In [62], Eq. (11) was extended involving a CIVF.

Theorem 3.10 *[62] Assume* $\mathfrak{f}$ *is a CIVF on* $[\Lambda,\Omega]$ *so that* $\mathfrak{f}(\Omega)\ge\mathfrak{f}(a_o)$, *for each* $a_o\in[\Lambda,\Omega]$, *then*

$$\mathfrak{f}\left(\Lambda+\Omega-\sum_{j=1}^n\varpi_j\,\beth_j\right)\supseteq\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\ominus_g\sum_{j=1}^n\varpi_j\,\beth_j.$$

For the rest, $\Gamma(.)$ is the Euler Gamma (see [63]). Also, $\ell,\beta>0$.

4. A H-H-Mercer type inequality using convex interval-valued functions through generalized fractional integrals (GFIs)

The concept of a convex function was explored over a century ago, leading to the development of a vast number of remarkable inequalities within convex theory. Among these, the H-H inequality stands out as one of the most well-known and widely applied. This inequality was first proposed by Hermite and Hadamard. The idea behind this inequality has inspired many mathematicians to explore and analyze classical inequalities through various convexity approaches.

The primary aim is to utilize the CIVF through a GFI operator to establish the H-H-Mercer type inclusion.

Theorem 4.1 *Let* $\Theta\in(0,1)$. *Let* $\mathfrak{f}:[\Lambda,\Omega]\to\mathbb{R}_I^+$ *be a CIVF so that* $\mathfrak{f}(\Theta)=[\underline{\mathfrak{f}}(\Theta),\overline{\mathfrak{f}}(\Theta)]$ *and* $\mathfrak{f}(\Omega)\ge\mathfrak{f}(\varpi_0)$, $\forall\varpi_0\in[\Lambda,\Omega]$. *Then*

$$\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\supseteq\frac{\ell^j\Gamma(j+1)}{2(\sigma-\varsigma)^{\ell j}} \tag{12}$$

$$\{ {}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{I}^{\ell}\mathfrak{f}(\Lambda+\Omega-\varsigma) + {}^{j}\mathfrak{I}^{\ell}_{(\Lambda+\Omega-\varsigma)}\mathfrak{f}(\Lambda+\Omega-\sigma)\}$$
$$\supseteq \frac{\mathfrak{f}(\Lambda+\Omega-\sigma)+\mathfrak{f}(\Lambda+\Omega-\varsigma)}{2} \supseteq (\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)) \ominus_g \frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.$$

Proof. We use a CIVF property on $\mathfrak{f}$, then for each $u,v \in [\Lambda,\Omega]$, we have

$$\mathfrak{f}\left(\Lambda+\Omega-\frac{u+v}{2}\right) = \mathfrak{f}\left(\frac{(\Lambda+\Omega-u)+(\Lambda+\Omega-v)}{2}\right)$$
$$\supseteq \frac{1}{2}\{\mathfrak{f}(\Lambda+\Omega-u)+\mathfrak{f}(\Lambda+\Omega-v)\}.$$

From the equations

$$\Lambda+\Omega-u = \Theta(\Lambda+\Omega-\varsigma)+(1-\Theta)(\Lambda+\Omega-\sigma)$$

and

$$\Lambda+\Omega-v = \Theta(\Lambda+\Omega-\sigma)+(1-\Theta)(\Lambda+\Omega-\varsigma),$$

for all $\varsigma,\sigma \in [\Lambda,\Omega]$ and $\Theta \in [0,1]$, we get

$$\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right) \supseteq \frac{1}{2}\{\mathfrak{f}(\Theta(\Lambda+\Omega-\varsigma)+(1-\Theta)(\Lambda+\Omega-\sigma))+\mathfrak{f}(\Theta(\Lambda+\Omega-\sigma)+(1-\Theta)(\Lambda+\Omega-\varsigma))\}. \quad (13)$$

Now, multiplying two sides of (13) by $\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}$ and integrating by inclusion with respect to (wrt) Θ on [0,1], we get

$$\begin{aligned}
\frac{1}{j\ell^{j}}\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right) &\supseteq \frac{1}{2}\{\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}(\Theta(\Lambda+\Omega-\varsigma)+(1-\Theta)(\Lambda+\Omega-\sigma))d\Theta \\
&\quad +\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}(\Theta(\Lambda+\Omega-\sigma)+(1-\Theta)(\Lambda+\Omega-\varsigma))d\Theta\} \\
&\supseteq \frac{1}{2}\{\frac{1}{(\sigma-\varsigma)^{\ell j}} \\
&\quad \times\int_{\Lambda+\Omega-\sigma}^{\Lambda+\Omega-\varsigma}\left(\frac{(\sigma-\varsigma)^{\ell}-((\Lambda+\Omega-\varsigma)-z)^{\ell}}{\ell}\right)^{j-1}\frac{\mathfrak{f}(z)}{((\Lambda+\Omega-\varsigma)-z)^{1-\ell}}dz+\frac{1}{(\sigma-\varsigma)^{\ell j}} \\
&\quad \times\int_{\Lambda+\Omega-\sigma}^{\Lambda+\Omega-\varsigma}\left(\frac{(\sigma-\varsigma)^{\ell}-(z-(\Lambda+\Omega-\sigma))^{\ell}}{\ell}\right)^{j-1}\frac{\mathfrak{f}(z)}{(z-(\Lambda+\Omega-\sigma))^{1-\ell}}dz\} \\
&\supseteq \frac{\Gamma(j)}{2(\sigma-\varsigma)^{\ell j}}\{ {}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{I}^{\ell}\,\mathfrak{f}(\Lambda+\Omega-\varsigma) + {}^{j}\mathfrak{I}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}(\Lambda+\Omega-\sigma)\}.
\end{aligned} \quad (14)$$

We proved the first inclusion in (12). Now, we show the second one. Having $\mathfrak{f}$ is a CIVF yields that

$$\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\varsigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\sigma\right)\right)\supseteq\Theta\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\left(1-\Theta\right)\mathfrak{f}\left(\Lambda+\Omega-\sigma\right), \tag{15}$$

and

$$\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\sigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\varsigma\right)\right)\supseteq\left(1-\Theta\right)\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\Theta\mathfrak{f}\left(\Lambda+\Omega-\sigma\right). \tag{16}$$

Adding (15) and (16), and using Jensen-Mercer inequality, one gets

$$\begin{aligned}
&\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\varsigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\sigma\right)\right)+\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\sigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\varsigma\right)\right)\\
&\supseteq\ \Theta F\left(\Lambda+\Omega-\varsigma\right)+\left(1-\Theta\right)\mathfrak{f}\left(\Lambda+\Omega-\sigma\right)+\left(1-\Theta\right)\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\Theta F\left(\Lambda+\Omega-\sigma\right)\\
&\supseteq\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\mathfrak{f}\left(\Lambda+\Omega-\sigma\right)\\
&\supseteq 2\{\mathfrak{f}\left(\Lambda\right)+\mathfrak{f}\left(\Omega\right)\}\ominus_g\{\mathfrak{f}\left(\varsigma\right)+\mathfrak{f}\left(\sigma\right)\}.
\end{aligned} \tag{17}$$

Now, we multiply two sides of (17) by $\left(\frac{1-\left(1-\Theta\right)^{\ell}}{\ell}\right)^{j-1}\left(1-\Theta\right)^{\ell-1}$ and we integrate by inequality wrt Θ on [0,1] to write

$$\begin{aligned}
&\int_0^1\left(\frac{1-\left(1-\Theta\right)^{\ell}}{\ell}\right)^{j-1}\left(1-\Theta\right)^{\ell-1}\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\varsigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\sigma\right)\right)d\Theta\\
&+\int_0^1\left(\frac{1-\left(1-\Theta\right)^{\ell}}{\ell}\right)^{j-1}\left(1-\Theta\right)^{\ell-1}\mathfrak{f}\left(\Theta\left(\Lambda+\Omega-\sigma\right)+\left(1-\Theta\right)\left(\Lambda+\Omega-\varsigma\right)\right)d\Theta\\
&\supseteq\{\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\mathfrak{f}\left(\Lambda+\Omega-\sigma\right)\}\int_0^1\left(\frac{1-\left(1-\Theta\right)^{\ell}}{\ell}\right)^{j-1}\left(1-\Theta\right)^{\ell-1}d\Theta\\
&\supseteq\{2\{\mathfrak{f}\left(\Lambda\right)+\mathfrak{f}\left(\Omega\right)\}\ominus_g\{\mathfrak{f}\left(\varsigma\right)+\mathfrak{f}\left(\sigma\right)\}\}\int_0^1\left(\frac{1-\left(1-\Theta\right)^{\ell}}{\ell}\right)^{j-1}\left(1-\Theta\right)^{\ell-1}d\Theta
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\Gamma\left(j\right)}{\left(\sigma-\varsigma\right)^{\ell j}}\{\,{}^{j}_{\left(\Lambda+\Omega-\sigma\right)}\mathfrak{I}^{\ell}\,\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)\ +{}^{j}\mathfrak{I}^{\ell}_{\left(\Lambda+\Omega-\varsigma\right)}\,\mathfrak{f}\left(\Lambda+\Omega-\sigma\right)\}\\
&\supseteq\frac{1}{j\ell^{j}}\{\mathfrak{f}\left(\Lambda+\Omega-\varsigma\right)+\mathfrak{f}\left(\Lambda+\Omega-\sigma\right)\}\\
&\supseteq\frac{1}{j\ell^{j}}\{2\{\mathfrak{f}\left(\Lambda\right)+\mathfrak{f}\left(\Omega\right)\}\ominus_g\{\mathfrak{f}\left(\varsigma\right)+\mathfrak{f}\left(\sigma\right)\}\}.
\end{aligned}$$

Dividing by 2 in above inclusion to get

$$\begin{aligned}
&\frac{\Gamma(j)}{2(\sigma-\varsigma)^{\ell j}}\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{I}^{\ell}\,\mathfrak{f}(\Lambda+\Omega-\varsigma)+{}^{j}\mathfrak{I}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}(\Lambda+\Omega-\sigma)\}\\
&\supseteq\frac{1}{2j\ell^{j}}\{\mathfrak{f}(\Lambda+\Omega-\varsigma)+\mathfrak{f}(\Lambda+\Omega-\sigma)\}\\
&\supseteq\frac{1}{2j\ell^{j}}\{2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_{g}\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\}\}.
\end{aligned}\tag{18}$$

Concatenating Equations (14) and (18), the proof is completed.

Consequently, we have the next results.

Corollary 4.2 *Using the same hypotheses as in Theorem 4.1 and with the condition* $\underline{\mathfrak{f}}(\Theta)=\overline{\mathfrak{f}}(\Theta)$, *we have*

$$\begin{aligned}
\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)&\le\frac{\ell^{j}\Gamma(j+1)}{2(\sigma-\varsigma)^{\ell j}}\times\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{I}^{\ell}\mathfrak{f}(\Lambda+\Omega-\varsigma)+{}^{j}\mathfrak{I}^{\ell}_{(\Lambda+\Omega-\varsigma)}\mathfrak{f}(\Lambda+\Omega-\sigma)\}\\
&\le\frac{\mathfrak{f}(\Lambda+\Omega-\varsigma)+\mathfrak{f}(\Lambda+\Omega-\sigma)}{2}\le(\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega))-\frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.
\end{aligned}$$

Corollary 4.3 *Using the same hypotheses as in Theorem 4.1 and with the condition* $\varsigma=\Lambda$ *and* $\sigma=\Omega$, *we have*

$$\mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right)\supseteq\frac{\ell^{j}\Gamma(j+1)}{2(\Omega-\Lambda)^{\ell j}}\{{}^{j}_{\Lambda}\mathfrak{I}^{\ell}\mathfrak{f}(\Omega)+{}^{j}\mathfrak{I}^{\ell}_{\Omega}\mathfrak{f}(\Lambda)\}\supseteq\frac{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)}{2}.$$

Theorem 4.4 *Let* $\mathfrak{f}:[\Lambda,\Omega]\to\mathbb{R}_I^{+}$ *be a CIVF so that* $\mathfrak{f}(\Theta)=[\underline{\mathfrak{f}}(\Theta),\overline{\mathfrak{f}}(\Theta)]$ *and* $\mathfrak{f}(\Omega)\ge\mathfrak{f}(\varpi_0)$, $\forall\varpi_0\in[\Lambda,\Omega]$. *Then*

$$\begin{aligned}
\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)&\supseteq\frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\ell^{j}\Gamma(j+1)\{{}^{j}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathfrak{I}^{\ell}\,\mathfrak{f}(\Lambda+\Omega-\varsigma)\\
&+{}^{j}\mathfrak{I}^{\ell}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\,\mathfrak{f}(\Lambda+\Omega-\sigma)\}\\
&\supseteq(\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega))\ominus_{g}\frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.
\end{aligned}\tag{19}$$

Proof. Using the CIVF, we have for all $u,v\in[\Lambda,\Omega]$,

$$\begin{aligned}
\mathfrak{f}\left(\Lambda+\Omega-\frac{u+v}{2}\right)&=\mathfrak{f}\left(\frac{(\Lambda+\Omega-u)+(\Lambda+\Omega-v)}{2}\right)\\
&\supseteq\frac{1}{2}\{\mathfrak{f}(\Lambda+\Omega-u)+\mathfrak{f}(\Lambda+\Omega-v)\}.
\end{aligned}$$

Using the equations

$$u=\frac{\Theta}{2}\varsigma+\frac{2-\Theta}{2}\sigma,$$

and

$$\upsilon = \frac{2-\Theta}{2}\varsigma + \frac{\Theta}{2}\sigma$$

for all $\varsigma, \sigma \in [\Lambda, \Omega]$ and $\Theta \in [0,1]$, one writes

$$\mathfrak{f}\left(\Lambda + \Omega - \frac{\varsigma+\sigma}{2}\right) \supseteq \frac{1}{2}\{\mathfrak{f}\left(\Lambda + \Omega - [\frac{\Theta}{2}\varsigma + \frac{2-\Theta}{2}\sigma]\right) + \mathfrak{f}\left(\Lambda + \Omega - [\frac{2-\Theta}{2}\varsigma + \frac{\Theta}{2}\sigma]\right)\}.$$

Now, we multiply both sides by $\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}$ and we integrate by inclusion wrt Θ on $[0,1]$ to obtain

$$\begin{aligned}
&\frac{1}{j\ell^{j}}\mathfrak{f}\left(\Lambda + \Omega - \frac{\varsigma+\sigma}{2}\right)\\
&\supseteq \frac{1}{2}\{\int_0^1 \left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda + \Omega - [\frac{\Theta}{2}\varsigma + \frac{2-\Theta}{2}\sigma]\right)d\Theta\\
&+\int_0^1 \left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda + \Omega - [\frac{2-\Theta}{2}\varsigma + \frac{\Theta}{2}\sigma]\right)d\Theta\}\\
&\frac{1}{j\ell^{j}}\mathfrak{f}\left(\Lambda + \Omega - \frac{\varsigma+\sigma}{2}\right)\\
&\supseteq \frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\Gamma(j)\{\,{}^{j}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathfrak{J}^{\ell}\,\mathfrak{f}\left((\Lambda+\Omega-\varsigma)\right)\\
&+\ {}^{j}\mathfrak{J}^{\ell}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\,\mathfrak{f}\left((\Lambda+\Omega-\sigma)\right)\}.
\end{aligned} \tag{20}$$

We showed the first inclusion (19). Now, we establish the second inclusion. Since $\mathfrak{f}$ is a CIVF, one has

$$\mathfrak{f}\left(\Lambda + \Omega - [\frac{\Theta}{2}\varsigma + \frac{2-\Theta}{2}\sigma]\right) \supseteq \mathfrak{f}(\Lambda) + \mathfrak{f}(\Omega) \ominus_g [\frac{\Theta}{2}\mathfrak{f}(\varsigma) + \frac{2-\Theta}{2}\mathfrak{f}(\sigma)]$$

and

$$\mathfrak{f}\left(\Lambda + \Omega - [\frac{2-\Theta}{2}\varsigma + \frac{\Theta}{2}\sigma]\right) \supseteq \mathfrak{f}(\Lambda) + \mathfrak{f}(\Omega) \ominus_g [\frac{2-\Theta}{2}\mathfrak{f}(\varsigma) + \frac{\Theta}{2}\mathfrak{f}(\sigma)].$$

Adding above equations, we get

$$\begin{aligned}
&\mathfrak{f}\left(\Lambda + \Omega - [\frac{\Theta}{2}\varsigma + \frac{2-\Theta}{2}\sigma]\right) + \mathfrak{f}\left(\Lambda + \Omega - [\frac{2-\Theta}{2}\varsigma + \frac{\Theta}{2}\sigma]\right)\\
&\supseteq 2\{\mathfrak{f}(\Lambda) + \mathfrak{f}(\Omega)\} \ominus_g \{\mathfrak{f}(\varsigma) + \mathfrak{f}(\sigma)\}.
\end{aligned} \tag{21}$$

We multiply by $\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}$ and integrate by inclusion wrt Θ on $[0,1]$ to get

$$
\{\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{\Theta}{2}\varsigma+\frac{2-\Theta}{2}\sigma]\right)d\Theta
$$

$$
+\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{2-\Theta}{2}\varsigma+\frac{\Theta}{2}\sigma]\right)d\Theta\}
$$

$$
\supseteq\left(2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_g\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\}\right)\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}d\Theta.
$$

Dividing by 2 in above inclusion, one finds

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\Gamma(j)\{{}^{j}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathfrak{J}^{\ell}\,\mathfrak{f}\left((\Lambda+\Omega-\varsigma)\right)+{}^{j}\mathfrak{J}^{\ell}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\,\mathfrak{f}\left((\Lambda+\Omega-\sigma)\right)\}\\
&\supseteq\frac{1}{2j\ell^{j}}\left(2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_g\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\}\right).
\end{aligned}
\tag{22}
$$

Concatenating Equations (20) and (22), we can get (19).
Consequently, we have the next results.

Corollary 4.5 *Using the same hypotheses as in Theorem 4.4 and with the condition* $\underline{\mathfrak{f}}(\Theta)=\overline{\mathfrak{f}}(\Theta)$, *we have*

$$
\begin{aligned}
\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)&\leq\frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\ell^{j}\Gamma(j+1)\{{}^{j}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathfrak{J}^{\ell}\,\mathfrak{f}(\Lambda+\Omega-\varsigma)\\
&\quad+{}^{j}\mathfrak{J}^{\ell}_{\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\,\mathfrak{f}(\Lambda+\Omega-\sigma)\}\\
&\leq\left(\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\right)-\frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.
\end{aligned}
$$

Remark 4.6 *If* $\varsigma=\Lambda$ *and* $\sigma=\Omega$ *in Corollary 4.5, then we obtain [64, Theorem 2.1].*

Remark 4.7 *If* $\varsigma=\Lambda$, $\sigma=\Omega$ *and* $\ell=1$ *in Corollary 4.5, then we have [65, Theorem 4].*

Theorem 4.8 *Let* $\mathfrak{f}:[\Lambda,\Omega]\to\mathbb{R}_I^+$ *be a CIVF such that* $\mathfrak{f}(\Theta)=[\underline{\mathfrak{f}}(\Theta),\overline{\mathfrak{f}}(\Theta)]$ *and* $\mathfrak{f}(\Omega)\geq\mathfrak{f}(\varpi_0)$, *for all* $\varpi_0\in[\Lambda,\Omega]$. *Then*

$$
\begin{aligned}
\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)&\supseteq\frac{\ell^{j}}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\Gamma(j+1)\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{J}^{\ell}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\\
&\quad+{}^{j}\mathfrak{J}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\}\\
&\supseteq\left(\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\right)\ominus_g\frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.
\end{aligned}
\tag{23}
$$

Proof. Using the property of a CIVF, one has

$$
\mathfrak{f}\left(\Lambda+\Omega-\frac{u+v}{2}\right)=\mathfrak{f}\left(\frac{(\Lambda+\Omega-u)+(\Lambda+\Omega-v)}{2}\right)\supseteq\frac{1}{2}\{\mathfrak{f}(\Lambda+\Omega-u)+\mathfrak{f}(\Lambda+\Omega-v)\},
$$

for each $u,v \in [\Lambda,\Omega]$. Letting

$$u = \frac{1-\Theta}{2}\varsigma + \frac{1+\Theta}{2}\sigma$$

and

$$v = \frac{1+\Theta}{2}\varsigma + \frac{1-\Theta}{2}\sigma$$

for all $\varsigma,\sigma \in [\Lambda,\Omega]$ and $\Theta \in [0,1]$, so we get

$$\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right) \supseteq \frac{1}{2}\{\mathfrak{f}\left(\Lambda+\Omega-[\frac{1-\Theta}{2}\varsigma+\frac{1+\Theta}{2}\sigma]\right)+\mathfrak{f}\left(\Lambda+\Omega-[\frac{1+\Theta}{2}\varsigma+\frac{1-\Theta}{2}\sigma]\right)\}.$$

We multiply by $\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}$ and integrate by inclusion wrt Θ on $[0,1]$ to write

$$\begin{aligned}
&\frac{1}{j\ell^{j}}\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\\
&\supseteq \frac{1}{2}\{\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\Theta-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{1-\Theta}{2}\varsigma+\frac{1+\Theta}{2}\sigma]\right)d\Theta\\
&+\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{1+\Theta}{2}\varsigma+\frac{1-\Theta}{2}\sigma]\right)d\Theta\}\\
&\supseteq \frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\\
&\times\{\int_{\Lambda+\Omega-\sigma}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\left(\frac{\left(\frac{2}{\sigma-\varsigma}\right)^{\ell}-\left(z-(\Lambda+\Omega-\sigma)\right)^{\ell}}{\ell}\right)^{j-1}\frac{\mathfrak{f}(z)}{\left(z-(\Lambda+\Omega-\sigma)\right)^{1-\Theta}}dz\\
&+\int_{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}^{\Lambda+\Omega-\varsigma}\left(\frac{\left(\frac{2}{\sigma-\varsigma}\right)^{\ell}-\left((\Lambda+\Omega-\varsigma)-z\right)^{\ell}}{\ell}\right)^{j-1}\frac{\mathfrak{f}(z)}{\left((\Lambda+\Omega-\varsigma)-z\right)^{1-\ell}}dz\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\frac{1}{j\ell^{j}}\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\\
&\supseteq \frac{\Gamma(j)}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\\
&\times\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{I}^{\ell}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)+{}^{j}\mathfrak{I}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\}.
\end{aligned}$$

Thus, the proof of the first inclusion is finished. Now, we use the Jensen-Mercer inequality to have

$$\mathfrak{f}\left(\Lambda+\Omega-[\frac{1-\Theta}{2}\varsigma+\frac{1+\Theta}{2}\sigma]\right)\supseteq\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\ominus_g[\frac{1-\Theta}{2}\mathfrak{f}(\varsigma)+\frac{1+\Theta}{2}\mathfrak{f}(\sigma)]$$

and

$$\mathfrak{f}\left(\Lambda+\Omega-[\frac{1+\Theta}{2}\varsigma+\frac{1-\Theta}{2}\sigma]\right)\supseteq\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\ominus_g[\frac{1+\Theta}{2}\mathfrak{f}(\varsigma)+\frac{1-\Theta}{2}\mathfrak{f}(\sigma)].$$

Adding above equations, we get

$$\begin{aligned}&\mathfrak{f}\left(\Lambda+\Omega-[\frac{1-\Theta}{2}\varsigma+\frac{1+\Theta}{2}\sigma]\right)+\mathfrak{f}\left(\Lambda+\Omega-[\frac{1+\Theta}{2}\varsigma+\frac{1-\Theta}{2}\sigma]\right)\\&\supseteq\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\ominus_g\left(\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\right).\end{aligned}\tag{25}$$

We multiply (25) by $\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}$ and integrate by inclusion wrt Θ on [0,1] to find

$$\begin{aligned}&\{\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{1-\Theta}{2}\varsigma+\frac{1+\Theta}{2}\sigma]\right)d\Theta\\&+\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}\mathfrak{f}\left(\Lambda+\Omega-[\frac{1+\Theta}{2}\varsigma+\frac{1-\Theta}{2}\sigma]\right)d\Theta\}\\&\supseteq(2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_g\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\})\int_0^1\left(\frac{1-(1-\Theta)^{\ell}}{\ell}\right)^{j-1}(1-\Theta)^{\ell-1}d\Theta\\&\Gamma(j)\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\times\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{J}^{\Theta}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)+{}^{j}\mathfrak{J}^{\Theta}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\}\\&\supseteq\frac{1}{j\Theta^{j}}(2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_g\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\}).\end{aligned}\tag{26}$$

Dividing by 2, one gets

$$\begin{aligned}&\frac{\Gamma(j)}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\times\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{J}^{\ell}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)+{}^{j}\mathfrak{J}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\}\\&\supseteq\frac{1}{2j\ell^{j}}(2\{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\}\ominus_g\{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)\}).\end{aligned}\tag{27}$$

Concatenating Equations (24) and (27), we can get (23).
Consequently, we have the next results.

Corollary 4.9 *Using the same hypotheses as in Theorem 4.8 and with the condition* $\underline{\mathfrak{f}}(\Theta)=\overline{\mathfrak{f}}(\Theta)$, *we have*

$$\begin{aligned}\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)&\le\frac{\ell^{j}}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\Gamma(j+1)\{{}^{j}_{(\Lambda+\Omega-\sigma)}\mathfrak{J}^{\ell}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\\&+{}^{j}\mathfrak{J}^{\ell}_{(\Lambda+\Omega-\varsigma)}\,\mathfrak{f}\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)\}\\&\le\left(\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)\right)-\frac{\mathfrak{f}(\varsigma)+\mathfrak{f}(\sigma)}{2}.\end{aligned}$$

Corollary 4.10 *Using same hypotheses as in Theorem 4.8 and with the condition $\varsigma = \Lambda$ and $\sigma = \Omega$, we have*

$$\mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right) \supseteq \frac{\ell^j}{2}\left(\frac{2}{\Omega-\Lambda}\right)^{\ell j} \Gamma(j+1)\{ {}_{\Lambda}^{j}\mathfrak{I}^{\ell}\, \mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right) + {}^{j}\mathfrak{I}_{\Omega}^{\ell}\, \mathfrak{f}\left(\frac{\Lambda+\Omega}{2}\right)\}$$
$$\supseteq \frac{\mathfrak{f}(\Lambda)+\mathfrak{f}(\Omega)}{2}.$$

5. Applications to matrices

Convexity and fractional calculus are widely used in many areas. These concepts have a vast range of intersting applications across various research fields, in fluid dynamics and optimization. More specifically, we aim to explore applications related to matrices. In [66], it was given that the function $\psi(\Upsilon) = \left\|\mathcal{G}^{\Upsilon}\mathcal{T}\mathcal{O}^{1-\Upsilon} + \mathcal{G}^{1-\Upsilon}\mathcal{T}\mathcal{O}^{\Upsilon}\right\|, \mathcal{G},\mathcal{O} \in M_n^{+}, \mathcal{T} \in M_n$, is convex for each Υ in [0,1].

Example 5.1 *Under considerations of above information and employing Theorem 4.1, we have*

$$\left\|\mathcal{G}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\frac{\varsigma+\sigma}{2})} + \mathcal{G}^{1-(\Lambda+\Omega-\frac{\varsigma+\sigma}{2})}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\right\| \supseteq \frac{\ell^j\Gamma(j+1)}{2(\sigma-\varsigma)^{\ell j}}$$
$$\times[{}_{\Lambda+\Omega-\sigma}^{j}\mathfrak{I}^{\ell} \left\|\mathcal{G}^{\Lambda+\Omega-\varsigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\varsigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\varsigma)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\varsigma}\right\|$$
$$+ {}^{j}\mathfrak{I}_{\Lambda+\Omega-\varsigma}^{\ell} \left\|\mathcal{G}^{\Lambda+\Omega-\sigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\sigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\sigma)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\sigma}\right\|]$$
$$\supseteq \frac{1}{2}\{\left\|\mathcal{G}^{\Lambda+\Omega-\varsigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\varsigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\varsigma)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\varsigma}\right\|$$
$$+\left\|\mathcal{G}^{\Lambda+\Omega-\sigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\sigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\sigma)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\sigma}\right\|\}$$
$$\supseteq \left\|\mathcal{G}^{\Lambda}\mathcal{T}\mathcal{O}^{1-\Lambda} + \mathcal{G}^{1-\Lambda}\mathcal{T}\mathcal{O}^{\Lambda}\right\| + \left\|\mathcal{G}^{\Omega}\mathcal{T}\mathcal{O}^{1-\Omega} + \mathcal{G}^{1-\Omega}\mathcal{T}\mathcal{O}^{\Omega}\right\|$$
$$\ominus_g \frac{1}{2}\{\left\|\mathcal{G}^{\varsigma}\mathcal{T}\mathcal{O}^{1-\varsigma} + \mathcal{G}^{1-\varsigma}\mathcal{T}\mathcal{O}^{\varsigma}\right\| + \left\|\mathcal{G}^{\sigma}\mathcal{T}\mathcal{O}^{1-\sigma} + \mathcal{G}^{1-\sigma}\mathcal{T}\mathcal{O}^{\sigma}\right\|\}.$$

Example 5.2 *Under considerations of above information and employing Theorem 4.4 and Example 5.1, we have*

$$\left\|\mathcal{G}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\mathcal{T}\mathcal{O}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)} + \mathcal{G}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\right\| \supseteq \frac{1}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j} \ell^j\Gamma(j+1)$$
$$\times[{}^{j}\mathfrak{I}_{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}^{\ell} \left\|\mathcal{G}^{\Lambda+\Omega-\sigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\sigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\sigma)}X\mathcal{O}^{\Lambda+\Omega-\sigma}\right\|$$
$$+ {}_{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}^{j}\mathfrak{I}^{\ell} \left\|\mathcal{G}^{\Lambda+\Omega-\varsigma}\mathcal{T}\mathcal{O}^{1-(\Lambda+\Omega-\varsigma)} + \mathcal{G}^{1-(\Lambda+\Omega-\varsigma)}\mathcal{T}\mathcal{O}^{\Lambda+\Omega-\varsigma}\right\|]$$
$$\supseteq \{ \left\|\mathcal{G}^{\Lambda}\mathcal{T}\mathcal{O}^{1-\Lambda} + \mathcal{G}^{1-\Lambda}\mathcal{T}\mathcal{O}^{\Lambda}\right\| + \left\|\mathcal{G}^{\Omega}\mathcal{T}\mathcal{O}^{1-\Omega} + \mathcal{G}^{1-\Omega}\mathcal{T}\mathcal{O}^{\Omega}\right\|\}$$
$$\ominus_g \frac{1}{2}\{\left\|\mathcal{G}^{\varsigma}\mathcal{T}\mathcal{O}^{1-\varsigma} + \mathcal{G}^{1-\varsigma}\mathcal{T}\mathcal{O}^{\varsigma}\right\| + \left\|\mathcal{G}^{\sigma}\mathcal{T}\mathcal{O}^{1-\sigma} + \mathcal{G}^{1-\sigma}\mathcal{T}\mathcal{O}^{\sigma}\right\|\}.$$

Example 5.3 *Under considerations of above information, employing Theorem 4.8 and Example 5.1, we have*

$$\begin{aligned}
&\left\|\mathcal{G}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\mathcal{TO}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}+\mathcal{G}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathcal{TO}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\right\| \\
&\supseteq \frac{\ell^{j}}{2}\left(\frac{2}{\sigma-\varsigma}\right)^{\ell j}\Gamma(j+1)\times[\,{}^{j}_{\Lambda+\Omega-\sigma}\mathfrak{I}^{\ell}\times\left\|\mathcal{G}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\mathcal{TO}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}+\mathcal{G}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathcal{TO}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\right\|+{}^{j}\mathfrak{I}^{\ell}_{\Lambda+\Omega-\varsigma} \\
&\times\left\|\mathcal{G}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\mathcal{TO}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}+\mathcal{G}^{1-\left(\Lambda+\Omega-\frac{\varsigma+\sigma}{2}\right)}\mathcal{TO}^{\Lambda+\Omega-\frac{\varsigma+\sigma}{2}}\right\| \\
&\supseteq \{\left\|\mathcal{G}^{\Lambda}\mathcal{TO}^{1-\Omega}+\mathcal{G}^{1-\Lambda}\mathcal{TO}^{\Lambda}\right\|+\left\|\mathcal{G}^{\Omega}\mathcal{TO}^{1-\Omega}+\mathcal{G}^{1-\Omega}\mathcal{TO}^{\Omega}\right\|\} \\
&\ominus_{g}\frac{1}{2}\{\left\|\mathcal{G}^{\varsigma}\mathcal{TO}^{1-\varsigma}+\mathcal{G}^{1-\varsigma}\mathcal{TO}^{\varsigma}\right\|+\left\|\mathcal{G}^{\sigma}\mathcal{TO}^{1-\sigma}+\mathcal{G}^{1-\sigma}\mathcal{TO}^{\sigma}\right\|\}.
\end{aligned}$$

6. Conclusions

Fractional calculus has a significant impact and yields more accurate results when analyzing computer models. It is extensively used in mathematical biology, engineering, applied mathematics, inequality theory and simulation. Many works from various scientific fields made a strong interest in fractional calculus. In our work:

- We investigated new variants inclusions of H-H-Mercer type for convex IVFs related to GFIs.
- We included several corollaries and remarks to enhance the reader's engagement and improve the overall quality of the work.
- We explored some meaningful applications related to matrices.

This work presented an intriguing and novel problem, offering aspiring researchers the opportunity to derive identical inequalities via various types of convexities within the framework of GFIs. The convexity theory can be leveraged to obtain a range of results in fields, like optimization theory, special functions, quantum mechanics and mathematical inequalities, while also encouraging further research across a wide array of pure and applied sciences.

Availability of data and material

All data generated or analysed during this work are included.

Authors' contributions

All authors read and approved the final work.

Competing interests

The authors declare that they have no competing interests.

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