



On certain properties of functions associated with a nonlinear operator

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Abstract

In this article, we develop a nonlinear operator $\mathfrak{T}_{\alpha,\gamma}^{\sigma}(f)(z)$:

$$\mathfrak{T}_{\alpha,\gamma}^{\sigma}(h)(z) := (1-\alpha)\sigma + \alpha\phi(z) - \frac{\gamma z\phi'(z)}{\phi(z)} - (1-\alpha-\gamma)\frac{z(\phi(z) + z\phi'(z))'}{\phi(z) + z\phi'(z)},$$

based on the functional $\phi(z) := \left(\frac{z}{f(z)}\right)^{\sigma} f'(z) : z \in \mathbb{E}$, the open unit disk, $\alpha, \gamma \in \mathbb{R}$, $\sigma \in [-1, 1]$ and find

conditions on the functional $\left(\frac{z}{f(z)}\right)^{\sigma} f'(z)$ so that it is a filtration. Moreover, we define a family $\mathcal{R}_{\sigma}(\alpha, \gamma)$ and study bounds on Fekete-Szegő functional $\mathcal{F}_f(\eta)$ alongwith some inclusions and different related results. These results can be further extended to symmetric, conjugate symmetric and other related setting in the present formulations.

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1. Introduction and Definitions

Simply, the term subordination in the w -plane is actually a generalization of some inequalities on the set $\mathbb{R}$. Obtaining information about a function from the properties of its derivatives is significant in both ways. Applications and extensions of differential and integral inequalities and subordination including the related theory have been developed in numerous fields of differential and integral equations as well as inequalities along with meromorphic and harmonic functions, operators, Banach spaces and others. Our intention is to incorporate these concepts to the recent trends in all such areas. The branching processes such as those observed in [1], and [2] have been the subject of the generation theory containing a single parameter semi-groups and these semi-groups related with the concept of filtration arose in the geometry of Banach spaces [3], finite-dimensional manifolds [4, 5], control and optimization theories [6], and one-dimensional complex analysis [7]. The foundations of several mathematical fields are presented by geometric function theory, which has been extensively studied since the early 1900s. For the past sixty years, a large number of mathematicians have been researching the theory of holomorphic mappings on complex spaces, with applications to functional analysis, quantum and classical physics, nonlinear analysis, and differential equations. Equations of motion, expressed in the abstract form of a dynamical system of a vector function f , which is a monotone operator on the underlying space, characterizing the state of the system under investigation, are used to convey the fundamental concepts of dynamics. As shown in [8, 9] and [10], analysts have focused on the studying of such systems

Take $\mathcal{H}(\mathbb{D}, \mathbb{C})$ as a set of holomorphic mappings from a domain $\mathbb{D}$ to $\mathbb{C}$. For $n \in \mathbb{N}$, we define

$$\mathcal{H}_n(\mathbb{E}, \mathbb{C}) := \left\{ f : f(z) = z + \sum_{j=n}^{\infty} a_{j+1} z^{j+1}, \quad z \in \mathbb{E} \right\}.$$

For $n = 1$, we have

$$\mathcal{H}(\mathbb{E}, \mathbb{C}) := \mathcal{A} := \left\{ f : f(z) = z + \sum_{j=1}^{\infty} a_{j+1} z^{j+1}, \quad z \in \mathbb{E} \right\}. \quad (1)$$

Here we note that

$$f(0) = f'(0) - 1 = 0,$$

which are the normalization conditions of f . We take

$$\mathcal{H}(\mathbb{E}, \mathbb{E}) := \mathcal{H}(\mathbb{E}). \quad (2)$$

Let $\mathcal{S} \subset \mathcal{A}$ be such that

$$\mathcal{S} := \{ f \in \mathcal{A} : f(z_1) = f(z_2) \Rightarrow z_1 = z_2 \}.$$

Let $\mathcal{P}$ be the family of analytic functions ψ such that $\operatorname{Re} \psi(z) > 0$ and

$$\psi(z) = 1 + \sum_{j=1}^{\infty} c_j z^j, \quad z \in \mathbb{E}.$$

The mapping $L_0 : L_0(z) = \frac{1+z}{1-z}$ is an example of the functions belonging to the class $\mathcal{P}$. Like the Koebe function, L_0 is an extremal function for the set $\mathcal{P}$. Related to the class $\mathcal{P}$, an other family $\mathcal{P}_\alpha$ for $0 \leq \alpha < 1$ can also be defined as: A function $\psi \in \mathcal{P}_\alpha$ if and only if there exists $\psi_1 \in \mathcal{P}$ such that $\psi(z) = (1-\alpha)\psi_1(z) + \alpha$. For $g(z)$ and $h(z)$ analytic in $\mathbb{E}$, $g(z)$ is subordinate to $h(z)$, if for a Schwarz

function $w(z)$, $g(z) = h(w(z))$, $z \in \mathbb{E}$. We denote it by $g(z) \prec h(z)$ as seen in [11]. A large number of subfamilies are related with the family $\mathcal{P}$ and its generalizations including the set $\mathcal{S}^*$ of starlike and a related set $\mathcal{C}$ of convex mappings. Moreover, both the subfamilies $\mathcal{S}_\alpha^*$ and $\mathcal{C}_\alpha$, $0 \leq \alpha < 1$, represent starlike and convex mappings of order α respectively. These families are further studied in such a way that the function ψ maps on to the right half plane or some specific plane regions. On the basis of these subfamilies, we develop more general and popular families of close-to-convex, spiral-like and other functions. For detail, see [11].

Much of the work in 20th century has been done in connection with the injective or univalent mappings. Various criterion for univalence are developed and other geometric characters of the image domain have been studied and considered in detail. For a function $f \in \mathcal{A}$ defined by (1), the differential conditions as seen in the inequalities $\operatorname{Re} f'(z) > 0$ or $|f'(z) - 1| \leq 1$, lead to the univalence of the mapping f in $\mathbb{E}$.

For $\eta \in \mathbb{C}$, the Fekete–Szegő functional $\mathcal{F}_f(\eta) := a_3 - \eta a_2^2$ involving coefficients of univalent analytic functions and found by Fekete and Szegő is related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete–Szegő problem. As seen in [12], the coefficient estimates over univalent functions were studied and it was proved that

$$|\mathcal{F}_f(\eta)| \leq 1 + 2 \exp\left(-\frac{2\eta}{1-\eta}\right).$$

In [13], it is seen for $f \in \mathcal{C}$

$$|\mathcal{F}_f(\eta)| \leq \max\left(\frac{1}{3}, |1-\eta|\right).$$

Also for $f \in \mathcal{S}_{\frac{1}{2}}^*$, we note that

$$|\mathcal{F}_f(\eta)| \leq \max\left(\frac{1}{2}, |1-\eta|\right). \quad (3)$$

These results are best possible. As it is seen in [14], we found that for a function $f \in \mathcal{A}_{\frac{1}{2}}$ such that

$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}$, we have

$$|\mathcal{F}_f(\eta)| \leq \max(1, |1-\eta|).$$

For a given $\varsigma > 0$, we introduce a family $\mathcal{R}_\varsigma$ of analytic functions such that $\sup_{f \in \mathcal{R}_\varsigma} |\mathcal{F}_f(\eta)| \leq \max(\varsigma, |1-\eta|)$. More specifically, we find interpolation $\{\mathcal{R}_\varsigma\}$, $\varsigma \in \left[\frac{1}{3}, 1\right]$ of the families $\mathcal{C}, \mathcal{A}_{\frac{1}{2}}$ such that $\mathcal{S}_{\frac{1}{2}}^* = \mathcal{R}_{\frac{1}{2}}$ and $\sup_{f \in \mathcal{R}_\varsigma} |\mathcal{F}_f(\eta)| \leq \max(\varsigma, |1-\eta|)$.

A mapping f such that

$$f(z) = \lim_{s \rightarrow 0^+} \frac{1}{s} [z - \varphi_s(z)], \quad s \geq 0$$

and φ_s is a unique solution of the problem (Cauchy) described by

$$\begin{cases} \frac{\partial}{\partial s}(\varphi_s(z)) + f(\varphi_s(z)) = 0, \\ \varphi_s(z) = z, s = 0 \end{cases}, \quad (4)$$

is termed as an infinitesimal generator of $\{\varphi_s, s \geq 0\} \subset \mathcal{H}(\mathbb{E})$, for details see [15, 16, 17]. A family $\mathcal{G}$ which is a set of all generators constitutes a semi-complete vector field. Another form of this set $\mathcal{G}$ studied in [7] and also found in [18] is rephrased as:

For $f \in \mathcal{H}(\mathbb{E}, \mathbb{C}) : f(z) \neq 0$, $f \in \mathcal{G}$, iff for $\tau \in \mathbb{E}$ and $\phi \in \mathcal{H}(\mathbb{E}, \mathbb{C})$ with $\operatorname{Re}\phi(z) \geq 0$, we have

$$f(z) = (\tau - z)(\bar{z}\tau - 1)\phi(z), z \in \mathbb{E}.$$

Particularly, $f \in \mathcal{G}$ iff, $\frac{f(z)}{z} \in \mathcal{P}$, but it is often difficult to establish. For example as in the case of the function $f(z) = z - 2\log(1-z)$, we see that the condition $\operatorname{Re}(f'(z)) = \operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0$ ensures that $f \in \mathcal{G}$. The condition $\operatorname{Re}(f'(z)) > 0$, proved in [16, 19] also implies the univalence of f and it is sufficient to assume that $f \in \mathcal{G}$. All generators (infinitesimal) are not univalent. Thus, if $f' \in \mathcal{P}$, then it does not mean that $f \in \mathcal{G}$.

A filtration of $\mathcal{G}$ is a set $\mathcal{D} = \{\mathcal{D}_t : t \in [c, d], \mathcal{D}_t \subseteq \mathcal{G}\}$, $c, d \in [-\infty, +\infty]$ and $\mathcal{D}_t \subseteq \mathcal{D}_s$, $t \leq s$. The filtration is strict if $\mathcal{D}_t \subset \mathcal{D}_s$, $t < s$.

We are focusing on developing a relationship between $\mathcal{F}_f(\eta)$ and the set $\mathcal{G}$. We also find a more generalized mechanism for $f \in \mathcal{A}$ such that $f \in \mathcal{G}$. We find a connection between the set $\mathcal{G}$ and the functional $\mathcal{F}_f(\eta)$ by determining $\{\mathcal{D}_\varsigma, \varsigma > 0\}$ such that $\sup_{f \in \mathcal{D}_\varsigma} |\mathcal{F}_f(\eta)| = \max(\varsigma, |1-\eta|)$. Next, we assume some general conditions for $f \in \mathcal{A}$ to be in $\mathcal{G}$ as below:

Definition 1.1 For $f \in \mathcal{A}$: let

$$\frac{F(z)}{z} = \phi(z) = \left(\frac{z}{f(z)}\right)^\sigma f'(z) = \left(\frac{zf'(z)}{f(z)}\right)^\sigma [f'(z)]^{1-\sigma} = [g_1(z)]^\sigma [g_2(z)]^{1-\sigma}$$

so that $\operatorname{Re}[g_1(z)] > 0$, $\operatorname{Re}[g_2(z)] > 0$, $-1 \leq \sigma \leq 1$, $z \in \mathbb{E} \setminus \{0\}$. Moreover, we let F has no zeros in $\mathbb{E}$.

Therefore, $F \in \mathcal{A}$ is in $\mathcal{G}$ iff $\operatorname{Re}\left[\frac{F(z)}{z}\right] = \operatorname{Re}\left[\left(\frac{z}{f(z)}\right)^\sigma f'(z)\right] > 0$.

By applying the Definition 1.1, we develop a non-linear differential operator

$$\mathfrak{I}_{\alpha, \gamma}^\sigma(f)(z) = (1-\alpha)\sigma + \varsigma\phi(z) - \frac{\gamma z\phi'(z)}{\phi(z)} - (1-\alpha-\gamma)\frac{z(\phi(z) + z\phi'(z))'}{\phi(z) + z\phi'(z)}, \quad (5)$$

based on the functional

$$\phi(z) = \frac{F(z)}{z} = \left(\frac{z}{f(z)}\right)^\sigma f'(z), f(z) \neq 0, z \in \mathbb{E} \setminus \{0\} \quad (6)$$

where $\alpha, \gamma \in \mathbb{R}$ and $\sigma \in [-1, 1]$, we determine conditions on the operator $\mathfrak{I}_{\alpha, \gamma}^\sigma(f)(z)$ so that F a filtration and establish sharp bounds on the modulus of $\mathcal{F}_f(\eta)$ over these filtration families along with some inclusions. We can further extend (2) for symmetric, conjugate symmetric and other related functions classes.

2. Preliminaries

We use some results given in the section below.

Lemma 2.1: Let ϕ be holomorphic such that $\operatorname{Re}[\gamma\phi(z) + \tau] > 0, \gamma, \tau \in \mathbb{C}, \gamma \neq 0$ and $\phi(z) = c + \sum_{j=1}^{\infty} c_j z^j, z \in \mathbb{E}$. Then the solution of

$$\psi(z) + \frac{z\psi'(z)}{\gamma\psi(z) + \tau} = \phi(z),$$

is holomorphic in $\mathbb{E}$ and satisfies $\operatorname{Re}[\gamma\psi(z) + \tau] > 0 : \psi(0) = c$.

For the detail of Lemma 2.1, see [20].

Lemma 2.2 For $\mathcal{H}(\mathbb{E})$ as defined in (2), $f \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ and a fixed $\tau \in \mathbb{R}$, we have $\operatorname{Re}f(z) > \tau, z \in \mathbb{E}$, iff, the functional ω defined by

$$\omega(z) = \frac{f(0) - f(z)}{2\tau - f(0) - f(z)} \in \mathcal{H}(\mathbb{E}). \quad (7)$$

Lemma 2.3 Suppose a mapping $\phi : \phi(0) = 1$ is analytic in $\mathbb{E}$. Let $\Omega \subset \mathbb{C}$ and a mapping $\Psi : \mathbb{C}^3 \times \mathbb{E} \rightarrow \mathbb{C}$ that satisfy

$$\Psi(i\rho, \mu, \theta + i\vartheta; z) \notin \Omega, \quad (z \in \mathbb{E}),$$

where $\rho, \mu, \theta, \vartheta \in \mathbb{R}, \mu \leq -\frac{1+\rho^2}{2}, \mu + \theta \leq 0$. If we take $\Psi(\phi(z), z\phi'(z), z^2\phi''(z); z) \in \Omega$, then it implies that $\operatorname{Re}\phi(z) > 0, z \in \mathbb{E}$.

For detail of Lemma 2.3 and Lemma 2.2, we refer to [21].

Lemma 2.4 If $\varphi(z) = \sum_{j=1}^{\infty} b_j z^j \in \mathcal{H}(\mathbb{E})$, then $|b_2| \leq 1 - |b_1|^2$ and $|b_2 - \eta b_1^2| \leq \max(1, |\eta|), \eta \in \mathbb{C}$.

The proof is seen in [13].

Lemma 2.5 Let $\varphi \in \mathcal{H}(\mathbb{E}) : |\varphi(z)| < 1$. If $|\varphi(z)|$ gets its maximum at z_0 , then

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = k, k \geq 1.$$

For detail, see [22].

3. Sufficient Conditions

In the theorem that follows, we intend to devise some conditions on $\mathfrak{I}_{\alpha, \gamma}^{\sigma}(f)$ such that $F \in \mathcal{G}$, where F given by (5).

Theorem 3.1 For $\alpha, \sigma, \gamma \in \mathbb{R}$, and $\sigma \in [-1, 1]$, take

$$\mathbb{D}_1 := \left\{ w : \sigma - \gamma - \zeta\sigma \geq x; (x - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2 \geq y^2; \alpha \geq 0 \right\}, \quad (8)$$

and

$$\mathbb{D}_2 := \left\{ w : x > \sigma - \gamma - \zeta\sigma; y^2 \leq (x - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2; \alpha < 0 \right\}. \quad (9)$$

For $\mathbb{D} \subset \mathbb{C}$ defined by

$$\mathbb{D} = \mathbb{C} \setminus \mathbb{D}_1, \alpha \geq 0 \text{ and } \mathbb{D} = \mathbb{C} \setminus \mathbb{D}_2, \alpha < 0, \quad (10)$$

if $F \in \mathcal{A} : \mathfrak{I}_{\alpha, \gamma}^\sigma(f)(\mathbb{E}) \subseteq \mathbb{D}$, then $F \in \mathcal{G}$, where F is given by (5).

Proof. From the mapping $f \in \mathcal{A} : \frac{F(z)}{z} = \left(\frac{z}{f(z)} \right)^\sigma f'(z)$ and $\phi(z) = \left(\frac{z}{f(z)} \right)^\sigma f'(z)$, such that $f(z) \neq 0$, $z \in \mathbb{E} \setminus \{0\}$, we observe that

$$\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = \sigma - \frac{z\phi'(z)}{\phi(z)},$$

and

$$\frac{z\phi'(z)}{1 + \sigma - \phi(z)} + \phi(z) = \sigma - \frac{(z^2\phi'(z))'}{(z\phi(z))'}.$$

In the context of (5), we have

$$\begin{aligned} \mathfrak{I}_{\alpha, \gamma}^\sigma(f)(z) &= \sigma\gamma + (1 - \alpha - \gamma)\sigma + \varsigma\phi(z) - \frac{\gamma z\phi'(z)}{\phi(z)} - (1 - \alpha - \gamma) \frac{(z^2\phi'(z))'}{(z\phi(z))'} \\ &= (1 - \alpha)\sigma + \varsigma\phi(z) - \frac{\gamma z\phi'(z)}{\phi(z)} - (1 - \alpha - \gamma) \frac{(z^2\phi'(z))'}{(z\phi(z))'}. \end{aligned}$$

Now to characterize the function $\phi(z)$, by choosing $r = \phi(z)$, $s = z\phi'(z)$, $t = z^2\phi''(z)$, we check the admissibility conditions as:

$$\Psi(r, s, t; z) = -\varepsilon \frac{2s + t}{s + r} - \frac{\gamma s}{r} + (1 - \alpha)\sigma + \varsigma r,$$

and

$$\Psi(\phi(z), z\phi'(z), z^2\phi''(z); z) = -\varepsilon \frac{(z^2\phi'(z))'}{(z\phi(z))'} - \frac{\gamma z\phi'(z)}{\phi(z)} + (1 - \alpha)\sigma + \varsigma\phi(z).$$

Using Lemma 2.3, we have to show that $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \in \mathcal{G}$ for $f \in \mathcal{A}$ or $\phi(\mathbb{E})$ is the right half plane. To meet this end, we prove that

$$\Psi(i\rho, \mu, \theta + i\vartheta; z) \notin \mathbb{D} \subset \mathbb{C}, \text{ when } \rho, \mu, \theta, \vartheta \in \mathbb{R}, \mu \leq -\frac{1}{2}(1 + \rho^2), \mu + \theta \leq 0. \quad (11)$$

We take

$$A = \operatorname{Re} \Psi(i\rho, \mu, \theta + i\vartheta; z) = (1 - \alpha)\sigma - \frac{(1 - \alpha - \gamma)}{\rho^2 + \mu^2} [(\theta + 2\mu)\mu + \vartheta\rho],$$

and

$$B = \operatorname{Im} \Psi(i\rho, \mu, \theta + i\vartheta; z) = \varsigma\rho + \frac{\gamma\mu}{\rho} - (1 - \alpha - \gamma) \frac{\vartheta\mu - (\theta + 2\mu)\rho}{\rho^2 + \mu^2}.$$

This implies that

$$\frac{B - \varsigma\rho - \frac{\gamma\mu}{\rho}}{A - (1 - \alpha)\sigma} = \frac{\vartheta\mu - (\theta + 2\mu)\rho}{(\theta + 2\mu)\mu + \vartheta\rho},$$

or we can write

$$\frac{B - \varsigma\rho - \frac{\gamma\mu}{\rho}}{A - (1 - \alpha)\sigma} = \frac{\mu}{\rho} - \frac{(\theta + 2\mu)(\rho^2 + \mu^2)}{\rho(\vartheta\rho + (\theta + 2\mu)\mu)}. \quad (12)$$

Moreover, if we let $m = A - (1 - \alpha)\sigma + \gamma$, then (12) becomes

$$B - \varsigma\rho - \frac{\gamma\mu}{\rho} = (m - \gamma) \frac{\mu}{\rho} - \frac{m - \gamma}{\rho} \frac{(\theta + 2\mu)(\rho^2 + \mu^2)}{(\vartheta\rho + (\theta + 2\mu)\mu)},$$

or

$$B = \rho\varsigma + m \frac{\mu}{\rho} - \frac{m - \gamma}{\rho} \frac{(\theta + 2\mu)(\rho^2 + \mu^2)}{(\vartheta\rho + (\theta + 2\mu)\mu)}.$$

Therefore, we have

$$B \neq \rho \left[\alpha + m \frac{\mu}{\rho^2} \right] = B_{\mu, \rho, \theta}(m), \text{ where } \rho \in \mathbb{R} \text{ and } \mu \leq -\frac{1}{2}(1 + \rho^2).$$

The condition (11) holds, if every point of $\mathbb{D}$, where is defined by (10) lies on the graph of $\frac{(\theta + 2\mu)(\rho^2 + \mu^2)}{(\vartheta\rho + (\theta + 2\mu)\mu)}$. We further study the range of the set $B_{\mu, \rho, \theta}(m)$. If $\alpha \geq 0, m > 0$, then from

$$y = \rho \left[\alpha + m \frac{\mu}{\rho^2} \right], \quad (13)$$

we have $\varsigma\rho^2 - \rho y + \mu m = 0$, which shows that

$$\rho = \frac{y \pm \sqrt{y^2 - 4\varsigma\mu m}}{2\varsigma}. \quad (14)$$

Also, by using (13) for $\mu \leq -\frac{(1 + \rho^2)}{2}$, we have

$$y = \varsigma\rho - m \frac{(1 + \rho^2)}{2\rho} \text{ or } \rho = \frac{-y \pm \sqrt{y^2 - (m - 2\varsigma)m}}{2(m - 2\varsigma)}.$$

Taking $y^2 - (m - 2\zeta)m \geq 0$ to write $y \geq \sqrt{(m - 2\zeta)m}$, where $(m - 2\zeta)m$ is assumed positive. Also

$$\begin{aligned} |B_{\mu,\rho,\theta}(A)| &= \left| \alpha - m \frac{1}{2\rho^2} (1 + \rho^2) \right| |\rho| \geq \left[\alpha - m \frac{(1 + \rho^2)}{2\rho^2} \right] |\rho| \geq \sqrt{(m - 2\zeta)m} \\ &= \sqrt{(A - (1 - \alpha)\sigma + \gamma - \alpha)^2 - \alpha^2}. \end{aligned}$$

Thus, $|B_{\mu,\rho,\theta}(A)| = \sqrt{(A - (1 - \alpha)\sigma + \gamma - \alpha)^2 - \alpha^2}$ for all reals, also if $y \in \mathbb{R}$, then we select ρ as given in (14), so (13) is satisfied. In the case $m \leq 0$, we write

$$|B_{\mu,\rho,\theta}(m)| \geq \sqrt{(m - 2\zeta)m} = \sqrt{(m - \alpha)^2 - \alpha^2} = \sqrt{(A - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2},$$

where we minimize $\left| \alpha - \frac{m(1 + \rho^2)}{2\rho^2} \right| |\rho|$ about ρ . In the case $\alpha \geq 0$ and $m \leq 0$, $|B_{\mu,\rho,\theta}(A)|^2$ takes all values say $X \geq (m - 2\zeta)m = (A - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2$. Thus for $m \leq 0$, $A \leq \sigma - \gamma - \zeta\sigma$ and the range set $\mathfrak{I}_{\alpha,\gamma}^\sigma(\mathbb{E})$ lies in the set $\mathbb{C} \setminus \mathbb{D}_1$ shown in (10), where $\mathbb{D}_1$ is given by (8). Also if we take $m \leq 0$; then we write

$$\begin{aligned} |B_{\mu,\rho,\theta}(m)| &= - \left[\alpha - m \frac{1 + \rho^2}{2\rho^2} \right] |\rho| \geq |\rho| \left(m \frac{1 + \rho^2}{2\rho^2} - \alpha \right) \geq \sqrt{(m - 2\zeta)m} \\ &= (A - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2. \end{aligned}$$

For $\alpha < 0$ and $m \geq 0$, $A > \sigma - \gamma - \zeta\sigma$ and $y^2 \geq (x - \sigma + \gamma + \alpha(\sigma - 1))^2 - \alpha^2$. In this case, the range set for the image domain of $\mathfrak{I}_{\alpha,\gamma}^\sigma(\mathbb{E})$ is the set $\mathbb{C} \setminus \mathbb{D}_2$ given by (9). For $\alpha < 0$, all graphs of $- \left[\alpha - m \frac{1 + \rho^2}{2\rho^2} \right] |\rho|$ are located in the set $\mathbb{C} \setminus \mathbb{D}_2$ shown in (10), where $\mathbb{D}_2$ is given by (9), $\rho \in \mathbb{R}$ and $\mu \leq -\frac{1}{2}(1 + \rho^2)$. In the context of Lemma 2.3 and above discussion we conclude that $\operatorname{Re}\phi(z) > 0$, that is $\phi(\mathbb{E})$ is the right half plane, hence $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) \in \mathcal{G}$ for $f \in \mathcal{A}$.

Remark 3.2 In view of $\mathbb{D}_1$, $\mathbb{D}_2$ and for $F \in \mathcal{A}$, $\sigma \in [-1, 1]$, $\alpha, \gamma \in \mathbb{R}$, $z \in \mathbb{E} \setminus \{0\}$, if $\alpha \geq 0$; $\frac{1}{2}(\alpha + \gamma) \geq \sigma - \gamma - \zeta\sigma : \gamma \geq \frac{1}{3}(2\sigma - \alpha(1 + 2\sigma))$, then, we have:

Corollary 3.3 For $\alpha, \gamma \in \mathbb{R}$, $\sigma \in [-1, 1]$, $F \in \mathcal{A}$ given by (5), and $z \in \mathbb{E} \setminus \{0\}$. If either $\gamma < \alpha$ for $\alpha \geq 0$ and $\operatorname{Re}\mathfrak{I}_{\alpha,\gamma}^\sigma(f)(z) > \sigma - \gamma - \zeta\sigma$, or $\gamma > \alpha$ for $\alpha < 0$ and $\operatorname{Re}\mathfrak{I}_{\alpha,\gamma}^\sigma(f)(z) < \sigma - \gamma - \zeta\sigma$, then $F \in \mathcal{G}$.

4. Maximization Problems

We introduce a new set $\mathcal{R}_\sigma(\alpha, \gamma)$ associated with the non linear operator $\mathfrak{I}_{\alpha,\gamma}^\sigma(f)$.

Definition 4.1: A mapping $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) \in \mathcal{A}$ given by (5) and $z \in \mathbb{E} \setminus \{0\}$ belongs to the family $\mathcal{R}_\sigma(\alpha, \gamma)$, if

$$\operatorname{Re} \mathfrak{I}_{\alpha, \gamma}^{\sigma}(f)(z) > \frac{1}{2}(\alpha + \gamma),$$

where $\gamma \geq \frac{1}{3}(2\sigma - \alpha(1 + 2\sigma))$ or

$$\operatorname{Re} \mathfrak{I}_{\alpha, \gamma}^{\sigma}(f)(z) > \frac{1}{3}(\sigma + \alpha(1 - \sigma)).$$

Thus, we observe or note that

For $\sigma = 1$ and $\alpha + \gamma \geq 2$, $\mathcal{R}_{\sigma}(\alpha, \gamma) = \mathbb{H}$. For $\alpha = 1$, $\gamma = 0$ and $\sigma \in [-1, 1]$, $\mathcal{R}_{\sigma}(1, 0) = \mathcal{G}$. For $\alpha = 0$ and $\sigma \in [-1, 1]$, $\mathcal{R}_{\sigma}(0, \gamma)$ is $(1 - \gamma)$ -convex of order $\frac{\gamma}{2}$, where $\gamma \geq \frac{2\sigma}{3}$.

Remark 4.2 For $\alpha \geq 0$, $\sigma \in [-1, 1]$, $\gamma \leq \frac{2\sigma - \alpha(1 + 2\sigma)}{3}$, we have $\frac{\alpha + \gamma}{2} \geq \sigma - \gamma - \zeta\sigma$ and $\mathcal{R}_{\sigma}(\alpha, \gamma) \subset \mathcal{G}$.

Subsequently, we determine conditions on α , γ and σ , so that $|\mathcal{F}_f(\eta)| \leq \max(\zeta, |1 - \eta|)$.

Theorem 4.3 Let $\alpha, \gamma \in \mathbb{R}$, $\sigma \in [-1, 1]$, $\alpha(1 - 2\sigma) - \gamma < -2\sigma$ and $7\alpha + 4\gamma < 6$, be such that $\gamma = \frac{2\sigma - \alpha(1 + 2\sigma)}{3}$ and $\frac{2\sigma + \alpha(1 - 2\sigma) - \gamma}{(3 - \sigma)(7\alpha + 4\gamma - 6)} > 0$. Then for $F(z) = \left(\frac{z}{f(z)}\right)^{\sigma} z f'(z) \in \mathcal{R}_{\sigma}(\alpha, \gamma): f \in \mathcal{A}$

$$|\mathcal{F}_f(\eta)| \leq \max\left(\frac{2\sigma + \alpha(1 - 2\sigma) - \gamma}{(3 - \sigma)(7\alpha + 4\gamma - 6)}, |1 - \eta|\right), \eta \in \mathbb{C}.$$

Proof. Let $F(z) = \left(\frac{z}{f(z)}\right)^{\sigma} z f'(z): f(z) \neq 0$ and $z \in \mathbb{E} \setminus \{0\}$ be in the set $\mathcal{R}_{\sigma}(\alpha, \gamma)$. Then $\mathfrak{I}_{\alpha, \gamma}^{\sigma}(f)(z)$ is obtained by using

$$\alpha z^{\sigma} [f(z)]^{-\sigma} f'(z) = \alpha + \alpha a_2(2 - \sigma)z + \alpha \left[\frac{\sigma^2 - 3\sigma}{2} a_2^2 + (3 - \sigma)a_3 \right] z^2 + \dots,$$

$$\gamma \varphi(z) = \gamma \left(\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right) = \gamma [\sigma + a_2(\sigma - 2)z + \{a_3(2\sigma - 6) - (\sigma - 4)\}z^2 + \dots],$$

and

$$\frac{zf'(z)}{1 + \sigma - \varphi(z)} + \varphi(z) = \sigma + 2a_2(\sigma - 2)z + \left[3a_3(2\sigma - 6) + (\sigma^2 - 7\sigma + 16)a_2^2 \right] z^2 + \dots$$

in (5) as seen below

$$\mathfrak{I}_{\alpha, \gamma}^{\sigma}(f)(z) = \sigma + (1 - \sigma)\alpha + (\sigma - 2)(2 - 3\alpha - \gamma)a_2z + [\eta_0 a_3 - \eta_1 a_2^2]z^2 + \dots \quad (15)$$

where

$$\eta_0 = (3 - \sigma)(7\alpha + 4\gamma - 6)$$

and

$$\eta_1 = -\alpha \left(\frac{\theta^2 + 3\sigma - 24}{2} \right) - \gamma(2\sigma - 8) - (\theta^2 - 7\sigma + 16).$$

Now consider that

$$\omega(z) = \frac{\mathfrak{I}_{\alpha,\gamma}^{\sigma}(f)(z) - \mathfrak{I}_{\alpha,\gamma}^{\sigma}(f)(0)}{\mathfrak{I}_{\alpha,\gamma}^{\sigma}(f)(0) + \mathfrak{I}_{\alpha,\gamma}^{\sigma}(f)(z) - (\alpha + \gamma)}.$$

By using (15), we see that

$$\omega(z) = \frac{(2-3\alpha-\gamma)(\sigma-2)a_2z}{2\sigma+\alpha-2\alpha\sigma-\gamma} + \left[\frac{(\eta_0a_3 - \eta_1a_2^2)a_3}{2\sigma+\alpha-2\alpha\sigma-\gamma} + \frac{(\sigma-2)^2(2-3\alpha-\gamma)^2a_2^2}{[2\sigma+\alpha-2\alpha\sigma-\gamma]^2} \right] z^2 + \dots$$

If we take $\omega(z) = \sum_{j=1}^{\infty} b_j z^j$, then we note that

$$\omega(z) = \frac{(2-3\alpha-\gamma)(\sigma-2)a_2z}{2\sigma+\alpha(1-2\sigma)-\gamma} + \left[\frac{(\eta_0a_3 - \eta_1a_2^2)a_3}{2\sigma+\alpha-2\alpha\sigma-\gamma} + \frac{(\sigma-2)^2(2-3\alpha-\gamma)^2a_2^2}{[2\sigma+\alpha(1-2\sigma)-\gamma]^2} \right] z^2 = \sum_{j=3}^{\infty} b_j z^j.$$

On comparison, we have

$$\begin{aligned} b_1 &= \frac{(\sigma-2)(2-3\alpha-\gamma)}{\alpha(1-2\sigma)-\gamma+2\sigma} a_2 = \tau_0 a_2, \\ b_2 &= \frac{\eta_0}{2\sigma+\alpha(1-2\sigma)-\gamma} \left[a_3 - \frac{\eta_1[\alpha(1-2\sigma)-\gamma+2\sigma] - (\sigma-2)^2(2-3\alpha-\gamma)^2}{[2\sigma+\alpha(1-2\sigma)-\gamma]\eta_0} a_2^2 \right] \\ &= \tau_1 [a_3 - \tau_2 a_2^2], \end{aligned}$$

where

$$\begin{aligned} \tau_0 &= \frac{(\sigma-2)(2-3\alpha-\gamma)}{\alpha(1-2\sigma)-\gamma+2\sigma}, \tau_1 = \frac{\eta_0}{\alpha(1-2\sigma)-\gamma+2\sigma}, \\ \tau_2 &= \frac{\eta_1[\alpha(1-2\sigma)-\gamma+2\sigma] - (\sigma-2)^2(2-3\alpha-\gamma)^2}{[2\sigma+\alpha(1-2\sigma)-\gamma]} \text{ and } \tau_3 = \tau_1 \tau_2. \end{aligned}$$

Thus, the Lemma 2.4 leads to

$$|b_2 - x b_1^2| = |\tau_1 [a_3 - \tau_2 a_2^2] - x \tau_0^2 a_2^2| = |\tau_1 a_3 - (\tau_3 + x \tau_0^2) a_2^2|,$$

or we see that

$$|b_2 - x b_1^2| = |\tau_1| \left| a_3 - \frac{(\tau_3 + x \tau_0^2)}{\tau_1} a_2^2 \right|.$$

If we denote $\eta = \frac{1}{\tau_1} (\tau_3 + x \tau_0^2)$, then we write

$$|a_3 - \eta a_2^2| = \frac{1}{|\tau_1|} |b_2 - x b_1^2|,$$

or

$$|a_3 - \eta a_2^2| = |\tau| |b_2 - x b_1^2|, \text{ where } |\tau| = \frac{1}{|\tau_1|},$$

and we can obtain a level set of functions

$$\mathcal{F}_\sigma(\alpha, \gamma) = \left[\frac{\eta_0}{\alpha(1-2\sigma) + 2\sigma - \gamma} \right]^{-1} = \frac{\alpha(1-2\sigma) + 2\sigma - \gamma}{(3-\sigma)(7\alpha + 4\gamma - 6)}.$$

By setting

$$\gamma = 2\sigma + \alpha(1-2\sigma) \text{ and } (3-\sigma)(7\alpha + 4\gamma - 6) = 0,$$

we obtain

$$\gamma = 2\sigma + \frac{(1-2\sigma)(6-8\sigma)}{11-8\sigma}.$$

Thus the corresponding level sets are containing rays that start from $\left(\frac{(6-8\sigma)}{11-8\sigma}, \frac{2\sigma(11-8\sigma) + (1-2\sigma)(6-8\sigma)}{11-8\sigma} \right)$ and are below the lines $2\sigma + \alpha(1-2\sigma) - \gamma = 0$ and $7\alpha + 4\gamma - 6 = 0$. This leads to the desired proof.

5. On Filtration Problems

Definition 5.1 Here we consider the set

$$\mathcal{R}_\sigma(0, \gamma) = \left\{ F(z) : \operatorname{Re} \mathfrak{I}_{0,\gamma}^\sigma(f)(z) > \frac{\gamma}{2} \text{ or } \operatorname{Re} \mathfrak{I}_{\alpha,\gamma}^\sigma(f)(z) > \frac{\sigma}{3}, \gamma \geq \frac{2\sigma}{3} \right\}, \quad (16)$$

of functions $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) : f \in \mathcal{A}, f(z) \neq 0, z \in \mathbb{E} \setminus \{0\}$ and $\operatorname{Re} \left\{ \varphi(z) + \frac{(1-\gamma)z\varphi'(z)}{1+\sigma-\varphi(z)} \right\} > \frac{\gamma}{2}$ for $\varphi(z) = \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}$.

Theorem 5.2 For $\gamma \neq 1$, we have

$$F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) \in \mathcal{R}_\sigma(0, \gamma) : f \in \mathcal{A} \Leftrightarrow g(z) = [1 + \sigma - f(z)]^{\gamma-1} \frac{[f(z)]^\sigma}{f'(z)} \in \mathcal{S}^*.$$

Proof. From (16), we see that

$$g(z) = [1 + \sigma - f(z)]^{\gamma-1} \frac{[f(z)]^\sigma}{f'(z)},$$

or we observe that

$$\frac{zg'(z)}{g(z)} = \varphi(z) + (1-\gamma)\sigma \frac{z\varphi'(z)}{1+\sigma-\varphi(z)}.$$

where $\varphi(z) = \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}$. Therefore, we reach the conclusion.

Next we assume that $\alpha = 0$ and $\sigma = 1$ in (5) and obtain

$$\mathfrak{S}_{0,\gamma}^1(f)(z) = 1 - \frac{\gamma z \phi'(z)}{\phi(z)} - (1-\gamma) \frac{z(\phi(z) + z\phi'(z))'}{\phi(z) + z\phi'(z)},$$

to consider the class

$$\mathcal{R}_1(0, \gamma) = \left\{ f \in \mathcal{A}, f(z) \neq 0, z \in \mathbb{E} \setminus \{0\} : \operatorname{Re} \mathfrak{S}_{0,\gamma}^1(f)(z) > \frac{1}{3} \right\},$$

or

$$\mathcal{R}_1(0, \gamma) = \left\{ f \in \mathcal{A}, f(z) \neq 0, z \in \mathbb{E} \setminus \{0\} : \operatorname{Re} \left\{ (1-\gamma) \frac{(zf'(z))'}{f'(z)} + \gamma \frac{zf'(z)}{f(z)} \right\} > \frac{\gamma}{2} \right\}.$$

The function f is $(1-\gamma)$ -convex function of order $\frac{\gamma}{2}$, as seen in [18] and others. We also get the following result as proved in [21]

Corollary 5.3 When $\gamma \neq 1$ is such that $\gamma < 2$, then

$$F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \in \mathcal{R}_1(0, \gamma) \Leftrightarrow g(z) = z[f'(z)]^{\frac{2-2\gamma}{2-\gamma}} \left[\frac{f(z)}{z} \right]^{\frac{2\gamma}{2-\gamma}} \in \mathcal{S}^*,$$

where $f \in \mathcal{A}$ is such that

$$f(z) = \left[\frac{1}{1-\gamma} \int_0^z u^{\frac{\gamma}{1-\gamma}} \left(\frac{g(u)}{u} \right)^{\frac{2-\gamma}{2-2\gamma}} du \right]^{1-\gamma}.$$

Theorem 5.4 Let $0 \leq \gamma \leq 1$ and $\sigma \in [-1, 1]$ and $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \in \mathcal{R}_\sigma(0, \gamma)$, $f(z) \neq 0$, $z \in \mathbb{E} \setminus \{0\}$

such that,

$$\operatorname{Re} \mathfrak{S}_{0,\gamma}^\sigma(f)(z) = \operatorname{Re} \left\{ \varphi(z) + \frac{(1-\gamma)z\phi'(z)}{1+\sigma-\varphi(z)} \right\} > \frac{\gamma}{2},$$

where $\varphi(z) = \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}$. Then $F \in \mathcal{S}_{\frac{1}{2}}^*$.

Proof. We use Lemma 2.2 to have $\operatorname{Re} \frac{zF'(z)}{F(z)} > \frac{1}{2}$, iff the function ω defined by (7) is such that

$$\omega(z) = 1 - \frac{1}{\varphi(z)}, \varphi(z) = \sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)],$$

which on differentiation leads to

$$\frac{1}{1-\omega(z)} = \varphi(z), \text{ and } 1 - \varphi(z) + \frac{zf''(z)}{f'(z)} = \frac{z\zeta'(z)}{1-\omega(z)}. \quad (17)$$

We make use of the Lemma 2.5 to have our desired task. We assume the mapping ω is not a holomorphic self-mapping. Then for $z_0 \in \mathbb{E} : |\omega(z)| < 1 : |z| < |z_0|$ and $|\omega(z_0)| = 1$. By Lemma 2.5, we have

$$\frac{z_0 \omega'(z_0)}{\omega(z_0)} = k \geq 1.$$

Using notation $\omega(z_0) = a + ib$, $a, b \in \mathbb{R} : a^2 + b^2 = 1$, we have

$$\operatorname{Re} \left\{ \frac{1}{1 - \omega(z_0)} \right\} = \frac{1 - a}{1 - 2a + (a^2 + b^2)} = \frac{1}{2}.$$

When $0 \leq \gamma \leq 1$ and $\sigma \in [-1, 1]$, then (17) yields

$$\begin{aligned} \operatorname{Re} \mathfrak{I}_{0,\gamma}^\sigma(f)(z_0) &= \operatorname{Re} \left\{ \frac{1}{2} \sigma + \frac{(1-\gamma)(-t\zeta(z_0))}{\left(\frac{1}{2} + \sigma\right)(1-\omega(z_0))^2} - \frac{\gamma}{2} \right\} \\ &= \frac{\sigma}{2} - \frac{k(1-\gamma)\sigma}{2(1+2\sigma)} - \frac{\gamma}{2} \leq 0, \end{aligned}$$

which is a contradiction and it leads the desired proof.

For the choices of $\alpha = 0$ and $\sigma = 1$, we have the following corollary already seen in [18, 21].

Corollary 5.5 *If $0 \leq \gamma \leq 1$, $\sigma = 1$ and $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z) \in \mathcal{R}_1(0, \gamma)$ that is,*

$$\operatorname{Re} \mathfrak{I}_{0,\gamma}^1 f(z) = \operatorname{Re} \left\{ \gamma \frac{z f'(z)}{f(z)} + (1-\gamma) \frac{(z f'(z))'}{f'(z)} \right\} > \frac{\gamma}{2},$$

then we have the assertion $\operatorname{Re} \frac{z f'(z)}{f(z)} > \frac{1}{2}$, that is $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Theorem 5.6 *For $\gamma < \gamma_1 \leq 1$, $\mathcal{R}_\sigma(0, \gamma) \subseteq \mathcal{R}_\sigma(0, \gamma_1)$.*

Proof. Let $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z) \in \mathcal{R}_\sigma(0, \gamma) : f \in \mathcal{A}, f(z) \neq 0, z \in \mathbb{E} \setminus \{0\}$. Then

$$\operatorname{Re} \mathfrak{I}_{0,\gamma}^\sigma(f)(z) > \frac{\gamma}{2} : \gamma \geq \frac{2\sigma}{3}, \sigma \in [-1, 1],$$

implies that there exists $\omega(z) \in \mathcal{H}(\mathbb{E})$ such that

$$\begin{aligned} \mathfrak{I}_{0,\gamma}^\sigma(f)(z) &= \frac{(1-\gamma)z\varphi'(z)}{1+\sigma-\varphi(z)} + \varphi(z); & \left(\varphi(z) = \sigma \frac{z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) \\ &= \frac{1}{1-\omega(z)} + \frac{(1-\gamma)\omega(z)}{[1-\omega(z)][(1+\sigma)(1-\omega(z))]} \\ &= \frac{1}{1-\omega(z)} \frac{[(1+\sigma)(1-\omega(z))] + (1-\gamma)\omega(z)}{(1+\sigma)(1-\omega(z))}. \end{aligned}$$

Also, we note that

$$\mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) - \frac{1}{2}\gamma = \frac{1}{1-\omega(z)} \frac{\left[(1+\sigma)(1-\omega(z)) \right] + (1-\gamma)\omega(z)}{(1+\sigma)(1-\omega(z))} - \frac{1}{2}\gamma,$$

or we can write

$$\mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) = \frac{2\left[(1+\sigma)(1-\omega(z)) \right] + 2(1-\gamma)\omega(z) - (1+\sigma)(1-\omega(z))^2}{2(1+\sigma)(1-\omega(z))^2} + \frac{1}{2}\gamma.$$

By using (7), we note that

$$\omega_{\gamma}(z) = -\frac{\mathfrak{I}_{0,\gamma}^{\sigma}(f)(0) - \mathfrak{I}_{0,\gamma}^{\sigma}(f)(z)}{\mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) - 2\tau + \mathfrak{I}_{0,\gamma}^{\sigma}(f)(0)}.$$

That is

$$\omega_{\gamma}(z) = \frac{\mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) - \sigma}{\mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) + \sigma - \gamma}, \quad (18)$$

and similarly for γ_1 , we consider that

$$\omega_{\gamma_1}(z) = \frac{\mathfrak{I}_{0,\gamma_1}^{\sigma}(f)(z) - \sigma}{\mathfrak{I}_{0,\gamma_1}^{\sigma}(f)(z) + \sigma - \gamma_1},$$

is holomorphic in $\mathbb{E}$ and zero at origin. We write (18) as

$$\left[\omega_{\gamma}(z) \right] \left[\sigma - \gamma + \mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) \right] = \mathfrak{I}_{0,\gamma}^{\sigma}(f)(z) - \sigma. \quad (19)$$

Similarly, we see that

$$\left[\omega_{\gamma_1}(z) \right] \left[\sigma - \gamma_1 + \mathfrak{I}_{0,\gamma_1}^{\sigma}(f)(z) \right] = \mathfrak{I}_{0,\gamma_1}^{\sigma}(f)(z) - \sigma. \quad (20)$$

Furthermore, we define the function

$$\mathfrak{I}_{0,\frac{1}{2}}^{\sigma}(f)(z) = \frac{\frac{1}{2}z \left(\frac{\sigma z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right)}{1 + \sigma - \left(\frac{\sigma z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right)} + \left(\sigma \frac{z f'(z)}{f(z)} - \frac{z f''(z)}{f'(z)} \right),$$

or

$$\phi(z) = \frac{z \left(\sigma \frac{z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right)}{1 + \sigma - \left(\sigma \frac{z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right)} + 2 \left(\sigma \frac{z f'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right).$$

where $\phi(z) = 2\mathfrak{I}_{0,\frac{1}{2}}^{\sigma}(f)(z)$. Then (19), implies that

$$(1-\gamma)[1-\omega_\gamma(z)]\phi(z) = (1-2\gamma)\sigma[1-\omega_\gamma(z)]\varphi(z) + \sigma[1+\omega_\gamma(z)] - \gamma\zeta_\gamma(z), \quad (21)$$

where $\varphi(z) = \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}$. Solving (21) to get the value of $\phi(z)$ as

$$\phi(z) = \frac{1-2\gamma}{1-\gamma}\varphi(z) + \frac{\sigma(1+\omega_\gamma(z))}{(1-\gamma)[1-\omega_\gamma(z)]} - \frac{\gamma\zeta_\gamma(z)}{(1-\gamma)[1-\omega_\gamma(z)]}. \quad (22)$$

Also equation (20) leads to

$$(1-\gamma_1)\phi(z) = (1-2\gamma_1)\varphi(z) + \frac{\sigma(1+\omega_{\gamma_1}(z))}{[1-\omega_{\gamma_1}(z)]} - \frac{\gamma_1\omega_{\gamma_1}(z)}{[1-\omega_{\gamma_1}(z)]}. \quad (23)$$

From (22) and (23), we observe that

$$\begin{aligned} & \left\{ \frac{\gamma_1}{1-\gamma_1} - \frac{\gamma}{1-\gamma} \right\} \varphi(z) + \frac{\sigma(1+\omega_\gamma(z))}{(1-\gamma)[1-\omega_\gamma(z)]} - \frac{\gamma\zeta_\gamma(z)}{(1-\gamma)(1-\omega_\gamma(z))} \\ &= \frac{\sigma(1+\omega_{\gamma_1}(z))}{(1-\gamma_1)[1-\omega_{\gamma_1}(z)]} - \frac{\gamma_1\omega_{\gamma_1}(z)}{(1-\gamma_1)(1-\omega_{\gamma_1}(z))}. \end{aligned} \quad (24)$$

On contrary, we assume the mapping $\omega_{\gamma_1}(z)$ does not a self mapping. Therefore, there exists $z_0 \in \mathbb{E}$: $|\omega_{\gamma_1}(z)| < 1$ for $|z| < |z_0|$ and $|\omega_{\gamma_1}(z_0)| = 1$. Substitute $z = z_0$ in (24) to have

$$\frac{\omega_\gamma(z)}{1-\omega_\gamma(z)} = \frac{-1}{2}, \quad \frac{1}{1-\omega_\gamma(z)} = \frac{1}{2},$$

and

$$\begin{aligned} & \left\{ \frac{\gamma_1}{1-\gamma_1} - \frac{\gamma}{1-\gamma} \right\} \operatorname{Re}\varphi(z_0) + \frac{\sigma}{1-\gamma} \operatorname{Re} \frac{(1+\omega_\gamma(z_0))}{(1-\omega_\gamma(z_0))} - \frac{\gamma}{1-\gamma} \operatorname{Re} \frac{\omega_\gamma(z_0)}{(1-\omega_\gamma(z_0))} \\ &= \frac{\sigma}{1-\gamma_1} \operatorname{Re} \frac{(1+\omega_{\gamma_1}(z_0))}{(1-\omega_{\gamma_1}(z_0))} - \frac{\gamma_1}{1-\gamma_1} \operatorname{Re} \frac{\omega_{\gamma_1}(z_0)}{(1-\omega_{\gamma_1}(z_0))}. \end{aligned}$$

or we note that

$$\left\{ \frac{\gamma_1}{1-\gamma_1} - \frac{\gamma}{1-\gamma} \right\} \operatorname{Re}\varphi(z_0) + 0 + \frac{\gamma}{2(1-\gamma)} > 0 + \frac{\gamma_1}{2(1-\gamma_1)}.$$

By using Theorem 5.4, we obtain

$$\left\{ \frac{\gamma_1}{1-\gamma_1} - \frac{\gamma}{1-\gamma} \right\} \varphi(z_0) + \frac{\gamma}{2(1-\gamma)} > \frac{1}{2} \left\{ \frac{\gamma_1}{1-\gamma_1} - \frac{\gamma}{1-\gamma} \right\} \sigma + \frac{\gamma}{2(1-\gamma)} = \frac{\gamma_1}{2(1-\gamma_1)},$$

which is a contradiction. Therefore, $\{\mathcal{R}_\sigma(0, \gamma), 1 \leq \sigma \leq 1\}$ is a filtration.

Corollary 5.7 For $\sigma = 1$ and $\gamma < \gamma_1 \leq 1, \mathcal{R}_1(0, \gamma) \subseteq \mathcal{R}_1(0, \gamma_1)$.

For detail, see [21].

6. Interpolation of $\mathcal{F}_f(\eta)$

We work out certain interpolation results regarding some estimate of $\mathcal{F}_f(\eta)$ given by (3).

Theorem 6.1 If the set $\{\mathcal{R}_\sigma(0, \gamma), \sigma \in [-1, 1]\}$ such that $\mathcal{R}_\sigma(0, \gamma) \subset \mathcal{G}$ is a filtration, then

$$\sup_{F \in \mathcal{R}_\sigma(0, \gamma)} |\mathcal{F}_f(\eta)| \leq \max \left(\frac{2\sigma - \gamma}{(3 - \sigma)(4\gamma - 6)}, |1 - \eta| \right), \eta \in \mathbb{C}$$

where $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z)$.

Proof. In the above Theorem 5.6, we have seen that $\{\mathcal{R}_\sigma(0, \gamma), \sigma \in [-1, 1]\}$ is a filtration of $\mathcal{G}$. We take $\alpha = 0$ and apply the Theorem 4.3 to reach the desired conclusion.

For $\alpha = 0$, $\sigma = 1$ and $\gamma \in \mathbb{R}$, we get the following special case as seen in [21].

Corollary 6.2 If the set $\{\mathcal{R}_1(0, \gamma)\} \subset \mathcal{G}$ so that $\mathcal{R}_1(0, \gamma) \subset \mathcal{S}_{\frac{1}{2}}^*$ is a filtration, then

$$\sup_{F \in \mathcal{R}_1(0, \gamma)} |\mathcal{F}_f(\eta)| \leq \max \left(\frac{2 - \gamma}{6 - 4\gamma}, |1 - \eta| \right),$$

where $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z)$ and $\eta \in \mathbb{C}$. In the context of (3), the sup is obtained for $\gamma = 0$ and $\gamma = 1$.

The above estimate is sharp when $\gamma \in (0, 1)$ and for $f_1, f_2 \in \mathcal{R}_1(0, \gamma)$ such that

$$\mathfrak{I}_{0, \gamma}^1(f_1)(z) = \left(1 - \frac{\gamma}{2} \right) \frac{1+z}{1-z} + \frac{\gamma}{2},$$

and

$$\mathfrak{I}_{0, \gamma}^1(f_2)(z) = \left(1 - \frac{\gamma}{2} \right) \frac{1+z^2}{1-z^2} + \frac{\gamma}{2}$$

are obtained from (5) and fulfill the Briot Bouquet equation. Hence, $|\mathcal{F}_{f_1}(\eta)| \leq |1 - \eta|$, $\eta \in \mathbb{C}$ and

$$|\mathcal{F}_{f_2}(\eta)| \leq \frac{2 - \gamma}{6 - 4\gamma}.$$

7. On Filtrations

For $f \in \mathcal{A}, f(z) \neq 0, z \in \mathbb{E} \setminus \{0\}$, we consider

$$\mathfrak{I}_{\alpha, 1-\alpha}^\sigma(f)(z) = \sigma(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{z}{f(z)} \right)^\sigma,$$

and

$$\mathcal{R}_\sigma(\alpha, 1 - \alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \mathfrak{I}_{\alpha, 1-\alpha}^\sigma(f)(z) > \frac{1}{2} \right\}.$$

We show that $\mathcal{S}_{\frac{1}{2}}^* \subset \mathcal{R}_{\alpha,1-\alpha}^\sigma$ for any $\alpha < 2$.

Theorem 7.1 For $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z) \in \mathcal{S}_{\frac{1}{2}}^*$, $\operatorname{Re}\left[\left(\frac{z}{f(z)}\right)^\sigma f'(z)\right] > \frac{1}{2}$, and $\mathcal{S}_{\frac{1}{2}}^* \subset \mathcal{R}_\sigma(\alpha, 1-\alpha)$, $\sigma \in [-1, 1]$. The sharp result holds for the mapping $z(1-z^n)^{-\frac{1}{\sigma}}$.

Proof. Assume that

$$\phi(z) = \left(\frac{z}{f(z)}\right)^\sigma f'(z) - 1,$$

and

$$\frac{z\phi'(z)}{1+\phi(z)} = \frac{zf''(z)}{f'(z)} - \sigma \frac{zf'(z)}{f(z)} + \sigma.$$

For $r = \phi(z)$, and $s = z\phi'(z)$, $\Psi(r, s) = \sigma - \frac{x}{(r+1)} : \Psi(i\rho, \mu) = \sigma - \frac{\mu}{(\rho^2+1)}$, if $\rho \in \mathbb{R}$ and $\mu \leq \left(\sigma - \frac{1}{2}\right)(\rho^2+1)$, then $\operatorname{Re}[\Psi(i\rho, \mu)] = \operatorname{Re}\left[1 - \frac{\mu}{(\rho^2+1)\sigma}\right] \leq 0$. Since $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z) \in \mathcal{S}_{\frac{1}{2}}^*$, so we have $\operatorname{Re}[\Psi(\phi, z\phi')(z)] > 0$. Therefore, $\operatorname{Re}\phi(z) > \frac{1}{2}$ or we can write $\operatorname{Re}\left[\left(\frac{z}{f(z)}\right)^\sigma f'(z)\right] > \frac{1}{2}$.

Theorem 7.2 For $\alpha < \alpha_1 \leq 1$, we have $\mathcal{R}_\sigma(\alpha, 1-\alpha) \subseteq \mathcal{R}_\sigma(\alpha_1, 1-\alpha_1)$.

Proof. We take $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z) \in \mathcal{R}_\sigma(\alpha, 1-\alpha)$ for $\sigma \in [-1, 1]$, $z \in \mathbb{E}$. Then $\operatorname{Re}\mathfrak{I}_{\alpha,1-\alpha}^\sigma(f)(z) > \frac{1}{2}$ shows that there exists a mapping $\psi_\alpha(z)$ in such a manner that from (7), we write

$$\psi_\alpha(z) = \frac{\mathfrak{I}_{\alpha,1-\alpha}^\sigma(f)(z) - \sigma - \alpha(1-\sigma)}{\mathfrak{I}_{\alpha,1-\alpha}^\sigma(f)(z) - (1-\alpha)(1-\sigma)}.$$

Then, from above functional equation, we see that

$$\begin{aligned} \psi_\alpha(z) & \left[(1-\alpha) \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) + \alpha z^\sigma [f(z)]^{-\sigma} f'(z) \right] - (1-\alpha)(1-\sigma) \\ & = \alpha z^\sigma [f(z)]^{-\sigma} f'(z) + (1-\alpha) \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) - \sigma - \alpha(1-\sigma). \end{aligned} \quad (25)$$

Also we have

$$2\mathfrak{I}_{\frac{1}{2}, \frac{1}{2}}^\sigma(f)(z) = \phi_1(z) = z^\sigma [f(z)]^{-\sigma} f'(z) - \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right), \quad (26)$$

By using (26) in (25), we observe that

$$\begin{aligned} \psi_\alpha(z) & \left[\alpha \left\{ z^\sigma [f(z)]^{-\sigma} f'(z) - \sigma \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right\} + \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} + (1-\alpha)(1-\sigma) \right] \\ & = \alpha \left\{ z^\sigma [f(z)]^{-\sigma} f'(z) - \left(\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right) \right\} + \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - \sigma - \alpha(1-\sigma), \end{aligned}$$

or

$$-\phi_1(z) = \frac{1}{\alpha} \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) + \frac{(1-\alpha)(1-\sigma)\psi_\alpha(z)}{\alpha(1-\psi_\alpha(z))} - \frac{\sigma + \alpha(1-\sigma)}{\alpha(1-\psi_\alpha(z))}. \quad (27)$$

Similarly in case of α_1 , we have

$$\begin{aligned} \psi_{\alpha_1}(z) & \left[(1-\alpha_1) \left(\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right) + \sigma + \alpha_1(1-\sigma) - 1 + \alpha_1 z^\sigma [f(z)]^{-\sigma} f'(z) \right] \\ & = \alpha_1 z^\sigma [f(z)]^{-\sigma} f'(z) - \sigma - \alpha_1(1-\sigma) + (1-\alpha_1) \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right), \end{aligned}$$

and

$$-\phi_1(z) = \frac{1}{\alpha_1} \left(\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right) + \frac{(1-\alpha_1)(1-\sigma)\psi_{\alpha_1}(z)}{\alpha_1(1-\psi_{\alpha_1}(z))} - \frac{\sigma + \alpha_1(1-\sigma)}{\alpha_1(1-\psi_{\alpha_1}(z))}. \quad (28)$$

Equating (27) and (28), we observe that

$$\begin{aligned} & \frac{1}{\alpha} \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) + \frac{(1-\alpha)(1-\sigma)\psi_\alpha(z)}{\alpha(1-\psi_\alpha(z))} - \frac{\sigma + \alpha(1-\sigma)}{\alpha(1-\psi_\alpha(z))} \\ & = \frac{1}{\alpha_1} \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln[f'(z)] \right) + \frac{(1-\alpha_1)(1-\sigma)\psi_{\alpha_1}(z)}{\alpha_1(1-\psi_{\alpha_1}(z))} - \frac{\sigma + \alpha_1(1-\sigma)}{\alpha_1(1-\psi_{\alpha_1}(z))}, \end{aligned}$$

or equivalently we have

$$\begin{aligned} & \left\{ \frac{1}{\alpha} - \frac{1}{\alpha_1} \right\} \left(-z \frac{d}{dz} \ln[f'(z)] + \sigma \frac{zf'(z)}{f(z)} \right) + \frac{(1-\alpha)(1-\sigma)\psi_\alpha(z)}{1-\psi_\alpha(z)} - \frac{\alpha(1-\sigma) + \sigma}{\alpha[1-\psi_\alpha(z)]} \\ & = \left[\frac{-\sigma - \alpha_1(1-\sigma)}{\alpha_1[1-\psi_{\alpha_1}(z)]} + \frac{(1-\sigma - \alpha_1(1-\sigma))\psi_{\alpha_1}(z)}{\alpha_1[1-\psi_{\alpha_1}(z)]} \right]. \end{aligned} \quad (29)$$

Obviously $\psi_{\alpha_1}(0) = 0$ and on contrary we suppose that the mapping $\psi_{\alpha_1}(z)$ is not a self-mapping. Then for $z_0 \in \mathbb{E} : |\psi_{\alpha_1}(z)| < 1 \ \forall \ |z| < |z_0|$ while $|\psi_{\alpha_1}(z_0)| = 1$. Substituting $z = z_0$ in (29) to have

$$\begin{aligned} & \left\{ \frac{1}{\alpha} - \frac{1}{\alpha_1} \right\} \operatorname{Re} \left(-z \frac{d}{dz} \ln[f'(z)] + \sigma \frac{zf'(z)}{f(z)} \right) - \operatorname{Re} \frac{\sigma + \alpha(1-\sigma)}{\alpha[1-\psi_\alpha(z)]} + \operatorname{Re} \frac{(1-\alpha)(1-\sigma)\psi_\alpha(z)}{1-\psi_\alpha(z)} \\ & > -\frac{1}{2\alpha_1}. \end{aligned}$$

Now applying Theorem 5.4, we see that

$$\left\{ \frac{1}{\alpha} - \frac{1}{\alpha_1} \right\} \operatorname{Re} \left(-z \frac{d}{dz} \ln [f'(z)] + \sigma \frac{zf'(z)}{f(z)} \right) - \operatorname{Re} \frac{\alpha(1-\sigma) + \sigma}{\alpha[1-\psi_\alpha(z)]} + \operatorname{Re} \frac{(1-\alpha)(1-\sigma)\psi_\alpha(z)}{1-\psi_\alpha(z)} < -\frac{1}{2\alpha_1}.$$

Hence, we find that

$$-\frac{1}{2\alpha_1} > \left\{ \frac{1}{\alpha} - \frac{1}{\alpha_1} \right\} \operatorname{Re} \left(\sigma \frac{zf'(z)}{f(z)} - z \frac{d}{dz} \ln [f'(z)] \right) - \frac{1}{2\alpha}.$$

This leads to the condition: $\operatorname{Re} \varphi(z) < \frac{1}{2}$ for $\varphi(z) = \sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}$. This is an obvious contradiction to our supposition. Therefore, the proof of our desired result is concluded.

The following theorems turns out that the set $\mathcal{R}_{\alpha,1-\alpha}^\sigma$ with $\frac{1}{2} \leq \alpha < 2$, constitutes some filtrations for certain generators along with sharp estimates over the quantity $|\mathcal{F}_f(\eta)|$.

Theorem 7.3 Let $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \in \mathcal{R}_\sigma(\alpha, 1-\alpha)$, $\frac{1}{2} \leq \alpha < 2$. Then $\mathcal{R}_\sigma(\alpha, 1-\alpha) \subset \mathcal{G}$. Furthermore, for $\frac{1}{2} \leq \alpha \leq 1$, each $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \in \mathcal{R}_\sigma(\alpha, 1-\alpha)$ describes a semi-group $\{\mathfrak{F}_s, s \geq 0\} \subset \mathcal{A}$ such that

$$|\mathfrak{F}_\alpha(z)| \leq e^{\frac{1-2\alpha}{2\alpha}t} |z|, \alpha = 1, \frac{1}{2} \text{ and } z \in \mathbb{B}.$$

Also the set $\{\mathcal{R}_\sigma(\alpha, 1-\alpha)\}$ is a filtration of $\mathcal{G}$ such that

$$\sup_{F \in \mathcal{R}_\sigma(\alpha, 1-\alpha)} |\mathcal{F}_f(\eta)| \leq \max \left\{ \frac{\alpha(2-2\sigma)-1+2\sigma}{(3-\sigma)(3\alpha+2)}, |1-\eta| \right\},$$

where $\eta \in \mathbb{C}$ and $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z)$.

Proof. Assume that $F(z) = \left(\frac{z}{f(z)} \right)^\sigma zf'(z) \notin \mathcal{G} : \operatorname{Re} \left\{ z^\sigma [f(z)]^{-\sigma} f'(z) \right\} > 0$ and take $\omega(z)$ as given by (7) with $\tau = 0$ in such a way that

$$\omega(z) = \frac{\varphi(z)-1}{\varphi(z)+1},$$

and

$$z^\sigma f'(z) [f(z)]^{-\sigma} = \frac{1+\omega(z)}{1-\omega(z)}, \quad \varphi(z) = z^\sigma f'(z) [f(z)]^{-\sigma}.$$

Moreover

$$-\frac{zf''(z)}{f'(z)} + \sigma \frac{zf'(z)}{f(z)} = \sigma - \frac{z\omega'(z)}{1+\omega(z)} - \frac{z\omega'(z)}{1-\omega(z)}.$$

For a fixed $z \in \mathbb{E}$, the mapping $\omega(z) \in \mathbb{E}$ is equivalent to $\operatorname{Re} z^\sigma [f(z)]^{-\sigma} f'(z) > 0$. In the perspective of our assumption, there is a point $z_0 \in \mathbb{E}$ so that $\operatorname{Re} z^\sigma [f(z)]^{-\sigma} f'(z) < 0 : |z| < |z_0|$ while $\operatorname{Re} \left(\frac{z_0}{f(z_0)} \right)^\sigma f'(z_0) = 0$ and hence, $|\omega(z_0)| = 1$. Therefore, by using Lemma 2.5, we find $k \geq 1$, such that $z_0 \omega'(z_0) = t\omega(z_0)$. A straightforward calculation leads to

$$\begin{aligned} \mathfrak{I}_{\alpha, 1-\alpha}^\sigma(f)(z_0) &= (1-\alpha) \left(\sigma \frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)} \right) + \alpha \left(\frac{z_0}{f(z_0)} \right)^\sigma f'(z_0) \\ &= \alpha \frac{\omega(z_0) + 1}{1 - \omega(z_0)} - \left(\frac{z_0 \omega'(z_0)}{1 - \omega(z_0)} + \frac{z_0 \omega'(z_0)}{\omega(z_0) + 1} - \sigma \right) (1-\alpha) \\ &= \alpha \frac{\omega(z_0) + 1}{1 - \omega(z_0)} - \left(\frac{t\omega(z_0)}{\omega(z_0) + 1} + \frac{t\omega(z_0)}{1 - \omega(z_0)} - \sigma \right) (1-\alpha). \end{aligned} \quad (30)$$

Using $\omega(z_0) = a + ib$, $a, b \in \mathbb{R} : a^2 + b^2 = 1$, we have

$$\operatorname{Re} \left\{ \frac{1}{1 - \omega(z_0)} \right\} = \frac{1 - a}{b^2 + a^2 - 2a + 1} = \frac{1}{2},$$

or we note that

$$\operatorname{Re} \frac{\omega(z_0)}{1 - \omega(z_0)} = \operatorname{Re} \frac{(1 - a + ib)(a + ib)}{b^2 + a^2 - 2a + 1} = \frac{-a^2 - b^2 + a}{2(1 - a)} = -\frac{1}{2}.$$

We also find that

$$\operatorname{Re} \left\{ \frac{1}{1 + \omega(z_0)} \right\} = \operatorname{Re} \left\{ \frac{1 + a - ib}{b^2 + a^2 + 2a + 1} \right\} = \frac{1}{2},$$

and finally we see that

$$\operatorname{Re} \frac{\omega(z_0)}{1 + \omega(z_0)} = \operatorname{Re} \frac{((1 + a) - ib)(a + ib)}{1 + 2a + b^2 + a^2} = \frac{1}{2}.$$

Using (30), we simplified that

$$\begin{aligned} \operatorname{Re} \mathfrak{I}_{\alpha, 1-\alpha}^\sigma(f)(z_0) &= \operatorname{Re} \left[\alpha \frac{1 + \omega(z_0)}{1 - \omega(z_0)} - \left(\frac{t\omega(z_0)}{1 + \omega(z_0)} - \sigma + \frac{t\omega(z_0)}{1 - \omega(z_0)} \right) (1-\alpha) \right] \\ &= (1-\alpha)\sigma. \end{aligned}$$

Since $F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) \in \mathcal{R}_\sigma(\alpha, 1-\alpha)$ and $\operatorname{Re} \mathfrak{I}_{\alpha, 1-\alpha}^\sigma(f)(z_0) = (1-\alpha)\sigma$, so we have $\alpha < \frac{1}{2}$. Thus

using (10), we see that $\mathcal{R}_\sigma(\alpha, 1-\alpha) \subset \mathcal{G}$, whenever $\alpha \geq \frac{1}{2}$. For the choice of $\alpha = 1$, we find that

$$\mathfrak{I}_{1,0}^\sigma(f)(z) = z^\sigma [f(z)]^{-\sigma} f'(z), \sigma \in [-1, 1].$$

and for the choice of $\alpha = \frac{1}{2}$, we see that

$$\mathfrak{I}_{\frac{1}{2}, \frac{1}{2}}^\sigma(f)(z) = \frac{1}{2} \left[\sigma \frac{zf'(z)}{f(z)} + z^\sigma [f(z)]^{-\sigma} f'(z) - \frac{zf''(z)}{f'(z)} \right].$$

In both of the above situations, we see that $f \in \mathcal{R}_\sigma(\alpha, 1-\alpha)$, which proves that the semi-group $\{\mathfrak{I}_s : s \geq 0\} \subset \mathcal{A}$, and it satisfies the problem given in (4). The function or mapping

$$F(z) = \left(\frac{z}{f(z)} \right)^\sigma z f'(z) \in \mathcal{A} \text{ given by}$$

$$\left[\lim_{s \rightarrow 0^+} \frac{z - \mathfrak{I}_s(z)}{s} \right]^\sigma = (F(z))^\sigma, \quad (s \geq 0)$$

with $\mathfrak{I}_s(z) = \exp(-as)z$, $a = \frac{1-2\zeta}{2\zeta} \in \mathbb{C}$, so that

$$\left[\lim_{s \rightarrow 0^+} \frac{z - \exp(-as)z}{s} \right]^\sigma = z^{-\sigma} \left[\lim_{s \rightarrow 0^+} \frac{1 - \exp(-as)}{s} \right]^\sigma = a^\sigma z^\sigma = (F(z))^\sigma, s \geq 0,$$

is clearly a generator (infinitesimal) for a single parameter set of semi-groups. Thus for each $z \in \mathbb{E}$, the problem in (4) is obviously take or assume a solution (unique) $\theta = \theta_s(z)$, $s \geq 0$ in such a way that $|\mathfrak{I}_s(z)| \leq e^{\frac{1-2\alpha}{2\alpha}t} |z|$, $\alpha = 1, \frac{1}{2}$ and $z \in \mathbb{E}$. So we assume $\frac{1}{2} < \alpha < 1$, and take

$$\varphi(z) = z^\sigma [f(z)]^{-\sigma} f'(z) - \left(\frac{1-2\alpha}{2\alpha} \right)^\sigma,$$

which can be written as

$$\frac{z\varphi'(z)}{\left(\frac{1-2\alpha}{2\alpha} \right)^\sigma + \varphi(z)} = \sigma - \sigma \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}.$$

This leads to

$$\begin{aligned} \mathfrak{I}_{\alpha, \delta}^\sigma(f)(z) &= \alpha z^\sigma [f(z)]^{-\sigma} f'(z) + \delta \left(\sigma \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right), \quad (\delta = 1 - \alpha) \\ &= \alpha \varphi(z) + \alpha \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma - \frac{z\varphi'(z)}{\frac{\varphi(z)}{\delta} + \frac{1}{\delta} \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma} + \delta \sigma^2 \\ &= \alpha \varphi(z) - \frac{z\varphi'(z)}{\frac{\varphi(z)}{\delta} + \frac{1}{\delta} \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma} + \delta \sigma^2 + \alpha \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma \\ &= \delta \sigma^2 + \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma + \alpha \varphi(z) - \frac{z\varphi'(z)}{\frac{1}{\delta} \left(\frac{\delta - \alpha}{2\alpha} \right)^\sigma + \frac{1}{\delta} \varphi(z)} \end{aligned}$$

From these simplifications and for the assumption $\delta = 1 - \alpha$, we note that

$$-\delta\sigma^2 - \left(\frac{\delta - \alpha}{2\alpha}\right)^\sigma + \mathfrak{I}_{\alpha,\delta}^\sigma(f)(z) = \alpha\varphi(z) - \frac{z\varphi'(z)}{\frac{1}{\delta}\varphi(z) + \frac{1}{\delta}\left(\frac{\delta - \alpha}{2\alpha}\right)^\sigma}.$$

In view of Lemma 2.1, we observe the solution $\varphi(z) = \phi(z)$ is holomorphic with $\operatorname{Re}\phi(z) = \operatorname{Re}z^\sigma [f(z)]^{-\sigma} f'(z) > \frac{1}{2} > 0$, $z \in \mathbb{E}$. Therefore, in the above situation, the operator $\mathfrak{I}_{\alpha,1-\alpha}^\sigma(f)(z)$ generates a single parameter semi-group and thus the set $\{\mathcal{R}_\sigma(\alpha, 1 - \alpha)\}$ is equivalent to $\mathcal{G}$. Also, if we take α a real, $f \in \mathcal{A}$, $\sigma \in [-1, 1]$ and $z \in \mathbb{E}$ such that

$$\frac{\alpha(2 - 2\sigma) - 1 + 2\sigma}{(3 - \sigma)(3\alpha + 2)} > 0.$$

Then $\sup_{F \in \mathcal{R}_{\alpha,1-\alpha}^\sigma} |\mathcal{F}_f(\eta)| \leq \max\left(\frac{\alpha(2 - 2\sigma) - 1 + 2\sigma}{(3 - \sigma)(3\alpha + 2)}, |1 - \eta|\right)$ for $\eta \in \mathbb{C}$ and $F(z) = \left(\frac{z}{f(z)}\right)^\sigma z f'(z)$, over the set $\mathcal{R}_\sigma(\alpha, 1 - \alpha)$.

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