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Characterizing two inclusive subfamilies of complex order defined by error functions and subordinate to horadam polynomials

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Abstract

In this paper, using the Error functions and subordinate to Horadam polynomials, we introduce two inclusive subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ of complex order. For functions in these subfamilies, we derive the estimations of the initial coefficients $|Q_2|$ and $|Q_3|$, as well as the Fekete-Szegö functional. Further, some related results are also obtained as corollaries and remark.

Mathematics Subject Classification (2010): 30C45

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1. Introduction and preliminaries

Based on recurrence relations, the Horadam polynomials are a family of polynomials that generalizes existing families, including the Fibonacci, Chebyshev, Pell, Pell-Lucas, and Lucas polynomials. In 1978, Australian mathematician Murray S. Klamkin Horadam introduced these polynomials, which bear his name.

Numerous intriguing characteristics are displayed by Horadam polynomials, which are related to number theory, algebraic geometry, and combinatorics, among other branches of mathematics.

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The following recurrence relation defined by Horadam [15, 16]:

$$h_{l+2} = \gamma h_{l+1} + \delta h_l, h_0 = \alpha, h_1 = \varepsilon, \alpha, \gamma, \delta, \varepsilon \in \mathbb{R}, l \in \mathbb{N}_0 = \mathbb{N} \cup 0.$$

For $l \in \mathbb{N} - \{1,2\}$, Horadam polynomials $h_l(y)$, is defined by:

$$h_l(y) = \gamma y h_{l-1}(y) + \delta h_{l-2}(y), \tag{1}$$

with

$$h_1(y) = \alpha, h_2(y) = \varepsilon y \text{and} h_3(y) = \gamma \varepsilon y^2 + \alpha\delta, \alpha, \gamma, \delta, \varepsilon \in \mathbb{R}, \tag{2}$$

where the generating function of the Horadam polynomials $h_l(y)$ is given as:

$$GH(y,\varsigma) = \sum_{l=1}^{\infty} h_l(y)\varsigma^{l-1} = \frac{\alpha + (\varepsilon - \alpha\gamma)y\varsigma}{1 - \gamma y \varsigma - \delta\varsigma^2}. \tag{3}$$

Remark 1.1 *The Horadam polynomials $h_l(y)$ yield a variety of polynomials for particular values of α, ε, γ and δ (see [15, 16]). For instance:*

1. At $\alpha = \varepsilon = \gamma = \delta = 1$, we get the Fibonacci polynomials $Fp_l(y)$;
2. At $\alpha = 2$ and $\varepsilon = \gamma = \delta = 1$, we get the Lucas polynomials $Lp_l(y)$;
3. At $\alpha = \varepsilon = 1$, $\gamma = 2$ and $\delta = -1$, we get the first kind of Chebyshev polynomials $Cp_l^1(y)$;
4. At $\alpha = 1$, $\varepsilon = \gamma = 2$ and $\delta = -1$, we get the second kind of Chebyshev polynomials $Cp_l^2(y)$;
5. At $\alpha = \delta = 1$ and $\varepsilon = \gamma = 2$, we get the Pell polynomials $Pp_l(y)$;
6. At $\alpha = \varepsilon = \gamma = 2$ and $\delta = 1$, we get the first kind of Pell-Lucas polynomials $PLp_l(y)$.

In complex analysis, error functions extend the ideas of quantifying deviations and measuring differences into the world of complex numbers. Statistics, probability science, modeling complicated variable physical processes, solving differential equations, and comprehending the behavior of analytic and non-analytic functions all depend on these functions (see [18, 19]).

The complex error function, commonly referred to as the Faddeeva function, is one of the most well-known instances. It generalizes the Gaussian error function to complex inputs. Because it sheds light on oscillatory and exponential behaviors in the complex plane, this function is essential to statistical physics, quantum mechanics, and wave propagation.

In systems affected by complex dynamics, error functions are especially helpful for estimating growth and decay rates, evaluating singularities, and modeling complex scenarios.

Elbert et al. [9] examined the characteristics of complementary error functions, whereas Coman [7] and Alzer [1] examined different characteristics and inequalities of error functions.

The definition of the error function erf, is (see [5], p. 297):

$$erf(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-\upsilon^2} d\upsilon = \frac{2}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{(-1)^l \varsigma^{2l+1}}{(2l+1)l!}, \qquad (\varsigma \in \mathbb{C}). \tag{4}$$

Additionally, the definition of the imaginary error function $erfi$, is

$$erfi(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{\upsilon^2} d\upsilon = \frac{2}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{\varsigma^{2l+1}}{(2l+1)l!}, \quad (\varsigma \in \mathbb{C}). \tag{5}$$

The following represents the generalized error function of (4) (see [5], p. 297):

$$erf_{\mu}(\varsigma) = \frac{\mu!}{\sqrt{\pi}} \int_0^{\varsigma} e^{\upsilon^{\mu}} d\upsilon = \frac{\mu!}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{\varsigma^{\mu l+1}}{(\mu l+1)l!}, \quad (\mu \in \mathbb{N}_0, \varsigma \in \mathbb{C}). \tag{6}$$

Also, the generalized imaginary error function of (5) given by:

$$erfi_{\mu}(\varsigma)=\frac{\mu!}{\sqrt{\pi}}\int_0^{\varsigma}e^{-\upsilon^{\mu}}d\upsilon=\frac{\mu!}{\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{(-1)^{l}\varsigma^{\mu l+1}}{(\mu l+1)l!},\quad (\mu\in\mathbb{N}_0,\varsigma\in\mathbb{C}). \tag{7}$$

Let Π represent the family of univalent and analytic functions F in the unit disk $\Delta=\{\varsigma\in\mathbb{C}:|\varsigma|<1\}$ normalized by $F(0)=F'(0)-1=0$. Therefore, each function $F\in\Pi$ has the following form (see [8]):

$$F(\varsigma)=\varsigma+\sum_{l=2}^{\infty}Q_l\varsigma^l,\ (\varsigma\in\Delta). \tag{8}$$

Therefore, the inverse of every function $F\in\Pi$ is F^{-1}, which is defined by

$$F^{-1}(F(\varsigma))=\varsigma \qquad (\varsigma\in\Delta)$$

and

$$F(F^{-1}(w))=w \qquad (|w|<\gamma_0(F);\gamma_0(F)\geq\frac{1}{4}),$$

where

$$G(w)\equiv F^{-1}(w)=w-Q_2w^2+(2Q_2^2-Q_3)w^3-(5Q_2^3-5Q_2Q_3+Q_4)w^4+\cdots. \tag{9}$$

Now, the subordination of analytic functions F_1 and F_2 (symbolizes by $F_1(\varsigma)\prec F_2(\varsigma)$ or $F_1\prec F_2$) if for all $\varsigma\in\Delta$ there exists a function Θ with $\Theta(0)=0$ and $|\Theta(\varsigma)|<1$; such that:

$$F_1(\varsigma)=F_2(\Theta(\varsigma)).$$

Also, if F_2 is univalent in Δ, then (see [21])

$$F_1(0)=F_2(0)\text{and}F_1(\delta)\subset F_2(\delta)\Leftrightarrow F_1(\varsigma)\prec F_2(\varsigma).$$

A function F, given by (8), belongs to the family BU of bi-univalent functions in Δ if both F and F^{-1} are univalent in Δ. For additional information about the family Ψ, see [17, 23, 25, 27].

The functions $erf_{\mu}(\varsigma)$ and $erfi_{\mu}(\varsigma)$ are obviously not members of the family Π. Thus, it seems sense to take into account the normalizations for these functions that Al-Hawary et al. suggested in [11] (see also, [13])

$$\mathcal{E}_{\mu}^{l}(\varsigma)=\frac{\sqrt{\pi}}{\mu!}\varsigma^{\left(1-\frac{1}{\mu}\right)}erf_{\mu}\left(\varsigma^{1/\mu}\right)=\varsigma+\sum_{l=2}^{\infty}\frac{(-1)^{l-1}}{((l-1)\mu+1)(l-1)!}\varsigma^{l},(\mu\in\mathbb{N},\varsigma\in\Delta), \tag{10}$$

and

$$E_{\mu}^{l}(\varsigma)=\frac{\sqrt{\pi}}{\mu!}\varsigma^{\left(1-\frac{1}{\mu}\right)}erfi_{\mu}\left(\varsigma^{1/\mu}\right)=\varsigma+\sum_{l=2}^{\infty}\frac{1}{((l-1)\mu+1)(l-1)!}\varsigma^{l},\qquad(\mu\in\mathbb{N}). \tag{11}$$

The convolution of two functions $F(\varsigma)=\varsigma+\sum_{l=2}^{\infty}Q_l\varsigma^l$ and $\gamma(\varsigma)=\varsigma+\sum_{l=2}^{\infty}D_l\varsigma^l$, define by

$$(F*\gamma)(\varsigma)=\varsigma+\sum_{l=2}^{\infty}Q_lD_l\varsigma^l,\ (\varsigma\in\Delta).$$

We define the function below using the convolution

$$EF_{\mu}^{l}(\varsigma) = F(\varsigma) * E_{\mu}^{l}(\varsigma) = \varsigma + \sum_{l=2}^{\infty} \frac{1}{((l-1)\mu+1)(l-1)!} Q_l \varsigma^l, (\mu \in \mathbb{N}).$$

Note that for $\mu = 2$ in (10), the normalization for Ramachandran et al. [24] is attained. The normalization for $\mu = 2$ in (11) is achieved for Mohammed et al. [22].

Bi-univalent functions associated with specific functions have been the subject of extensive investigation (see [2, 4, 12-14]), and several well-known families include the Jacobi, Laguerre, Legendre, Hermite, Chebyshev and many other functions (see [3, 26, 28, 29]) .

Ezrohi [10] introduced the family

$$\mathcal{U}(\zeta) = \left\{ F : F \in \Pi \text{ and } Re\left\{F'(\varsigma)\right\} > \zeta, \quad (\varsigma \in \Delta; 0 \le \zeta < 1) \right\}.$$

Also, Chen [6] introduced the family

$$\mathcal{ST}(\zeta) = \left\{ F : F \in \Pi \text{ and } Re\left\{\frac{F(\varsigma)}{\varsigma}\right\} > \zeta, \quad (\varsigma \in \Delta; 0 \le \zeta < 1) \right\}.$$

Motivated by the previous two families, in this work, we introduced the comprehensive subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ of complex order using error functions and subordinate to Horadam polynomials. The upper bounds of the coefficients $|Q_2|$, $|Q_3|$ and the functional $|Q_3 - \varepsilon Q_2^2|$ are estimated for these subfamilies.

2. Coefficient bounds for the inclusive subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$

The definitions of the inclusive subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ of complex order using error functions and Horadam polynomials are given first in this section.

Definition 2.1: *Let $\eta > 0$, $\rho \in \mathbb{R}$, $\eta + i\rho \neq 0$, $\varsigma, w \in \mathbb{C}$ and the function $GH(y,\varsigma)$ is given by (3). A function $F \in BU$ given by (8) is said to be in the subfamily $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ if the subsequent subordinations are met:*

$$\left[\left(EF_{\mu}^{i}(\varsigma)\right)'\right]^{\eta+i\rho} \prec GH(y,\varsigma) + 1 - \alpha \tag{12}$$

and

$$\left[\left(EG_{\mu}^{i}(w)\right)'\right]^{\eta+i\rho} \prec GH(y,w) + 1 - \alpha. \tag{13}$$

Definition 2.2 *Let $\eta > 0$, $\rho \in \mathbb{R}$, $\eta + i\rho \neq 0$, $\varsigma, w \in \mathbb{C}$ and the function $GH(y,\varsigma)$ is given by (3). A function $F \in BU$ given by (8) is said to be in the subfamily $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ if the subsequent subordinations are met:*

$$\left[\frac{EF_{\mu}^{i}(\varsigma)}{\varsigma}\right]^{\eta+i\rho} \prec GH(y,\varsigma) + 1 - \alpha \tag{14}$$

and

$$\left[\frac{EG_{\mu}^{i}(w)}{w}\right]^{\eta+i\rho} \prec GH(y,w) + 1 - \alpha. \tag{15}$$

Lemma 2.3 *([30]) Let* $\sigma_1,\sigma_2\in\mathbb{R}$ *and* $\tau_1,\tau_2\in\mathbb{C}$. *If* $|\tau_1|<\hbar$ *and* $|\tau_2|<\hbar$, *then*

$$|(\sigma_1+\sigma_2)\tau_1+(\sigma_1-\sigma_2)\tau_2|\le\begin{cases}2|\sigma_1|\hbar \text{ for } |\sigma_1|\ge|\sigma_2|,\\ 2|\sigma_2|\hbar \text{ for } |\sigma_1|\le|\sigma_2|.\end{cases}$$

Lemma 2.4 *([20]) If* $C(\varsigma)=1+Q_1\varsigma+Q_2\varsigma^2+\cdots\in\Upsilon$, $\varsigma\in\Delta$, *then there exist some* V,L *with* $|V|\le1$, $|L|\le1$, *such that*

$$2Q_2=Q_1^2+V(4-Q_1^2)\text{ and}4Q_3=Q_1^3+2Q_1(4-Q_1^2)V-(4-Q_1^2)Q_1V^2+2(4-Q_1^2)(1-|V|^2)L. \tag{16}$$

Theorem 2.5 *Let* $F\in BU$ *given by (8). If* $F\in AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$. *Then*

$$|Q_2|\le\min\left\{\frac{|\varepsilon y|(\mu+1)}{2|\eta+i\rho|},\frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\varepsilon^2y^2\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{8(\gamma\varepsilon y^2+\alpha\delta)(\eta+i\rho)^2}{(\mu+1)^2}\right|}}\right\},$$

$$|Q_3|\le\min\left\{\frac{\varepsilon^2y^2(\mu+1)^2}{4|\eta+i\rho|^2}+\frac{2|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|},\right.$$

$$\left.\frac{2|\varepsilon y|^3}{\left|\varepsilon^2y^2\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{8(\gamma\varepsilon y^2+\alpha\delta)(\eta+i\rho)^2}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|}\right\}$$

and

$$|Q_3-\rho Q_2^2|\le\begin{cases}\dfrac{4|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|} & |1-\rho|<\dfrac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2},\\[2ex] \dfrac{\varepsilon^2y^2(\mu+1)^2|1-\rho|}{|\eta+i\rho|^2} & |1-\rho|\ge\dfrac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2}.\end{cases} \tag{17}$$

Proof. Let $F\in AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$. So, from (12) and (13), we can write

$$\left[\left(EF_\mu^i(\varsigma)\right)'\right]^{\eta+i\rho}=GH(y,T(\varsigma))+1-\alpha,\varsigma\in\Delta$$

and

$$\left[\left(EG_\mu^i(w)\right)'\right]^{\eta+i\rho}=GH(y,V(w))+1-\alpha,\ w\in\Delta,$$

where the analytic functions T and V are of the form:

$$T(\varsigma)=\upsilon_1\varsigma+\upsilon_2\varsigma^2+\upsilon_3\varsigma^3+\cdots,$$

and

$$V(w)=t_1w+t_2w^2+t_3w^3+\cdots,$$

such that $T(0)=V(0)=0$ and $|T(\varsigma)|<1, |V(\varsigma)|<1$ for $\varsigma, w\in\Delta$.

Thus we have

$$\left[\left(EF_\mu^i(\varsigma)\right)'\right]^{\eta+i\rho}=1+h_2(y)\upsilon_1\varsigma+\left(h_2(y)\upsilon_2+h_3(y)\upsilon_1^2\right)\varsigma^2+\cdots,\varsigma\in\Delta. \tag{18}$$

and

$$\left[\left(EG_\mu^i(w)\right)'\right]^{\eta+i\rho}=1+h_2(y)t_1w+\left(h_2(y)t_2+h_3(y)t_1^2\right)w^2+\cdots,w\in\Delta, \tag{19}$$

such that

$$|\upsilon_l|\le 1 \text{and} |t_l|\le 1, l\in\mathbb{N}. \tag{20}$$

From (18) and (19), we have

$$\frac{2(\eta+i\rho)}{\mu+1}Q_2=h_2(y)\upsilon_1, \tag{21}$$

$$\frac{2(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}Q_2^2+\frac{3(\eta+i\rho)}{2(2\mu+1)}Q_3=h_2(y)\upsilon_2+h_3(y)\upsilon_1^2, \tag{22}$$

$$-\frac{2(\eta+i\rho)}{\mu+1}Q_2=h_2(y)t_1, \tag{23}$$

and

$$\left[\frac{3(\eta+i\rho)}{2\mu+1}+\frac{2(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right]Q_2^2-\frac{3(\eta+i\rho)}{2(2\mu+1)}Q_3=h_2(y)t_2+h_3(y)t_1^2. \tag{24}$$

From (21) and (23) it follows that

$$\upsilon_1=-t_1 \tag{25}$$

and

$$\frac{8\left(\eta+i\rho\right)^2}{(\mu+1)^2}Q_2^2=\left(h_2(y)\right)^2\left(\upsilon_1^2+t_1^2\right). \tag{26}$$

Add the equations (22) and (24), then substituting the value of $\upsilon_1^2+t_1^2$ from (26), we get

$$\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}-\frac{8h_3(y)\left(\eta+i\rho\right)^2}{\left(h_2(y)\right)^2(\mu+1)^2}\right)Q_2^2=h_2(y)(\upsilon_2+t_2). \tag{27}$$

Using the triangle inequality for the equations (21) and (27), and using (20), we obtain respectively:

$$|Q_2| \le \frac{|\varepsilon y|(\mu+1)}{2|\eta+i\rho|} \text{ and } |Q_2| \le \frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\varepsilon^2 y^2\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)(\eta+i\rho)^2}{(\mu+1)^2}\right|}}.$$

Also, if we subtract (24) from (22), we have

$$\frac{3(\eta+i\rho)}{2\mu+1}\left(Q_3-Q_2^2\right)=h_2(y)\left(\upsilon_2-t_2\right)+h_3(y)\left(\upsilon_1^2-t_1^2\right). \tag{28}$$

In view of (25), the equation (28) becomes

$$Q_3=Q_2^2+\frac{h_2(y)\left(\upsilon_2-t_2\right)(2\mu+1)}{3(\eta+i\rho)}. \tag{29}$$

Using (25) and (26), the equation (29) becomes

$$Q_3=\frac{\left(h_2(y)\right)^2(\mu+1)^2\upsilon_1^2}{4\left(\eta+i\rho\right)^2}+\frac{h_2(y)\left(\upsilon_2-t_2\right)(2\mu+1)}{3(\eta+i\rho)}. \tag{30}$$

Using the triangle inequality and (20) for equation (30), we obtain

$$|Q_3| \le \frac{\varepsilon^2 y^2(\mu+1)^2}{4|\eta+i\rho|^2}+\frac{2|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|}.$$

Similarly, using of (27) in (29), we have

$$Q_3=\frac{\left(h_2(y)\right)^3(\upsilon_2+t_2)}{\left(h_2(y)\right)^2\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{8h_3(y)\left(\eta+i\rho\right)^2}{(\mu+1)^2}}+\frac{h_2(y)\left(\upsilon_2-t_2\right)(2\mu+1)}{3(\eta+i\rho)} \tag{31}$$

Using the triangle inequality and (20) for (31), we obtain

$$|Q_3| \le \frac{2|\varepsilon y|^3}{\left|\varepsilon^2 y^2\left(\frac{3(\eta+i\rho)}{2\mu+1}+\frac{4(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)(\eta+i\rho)^2}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|}.$$

Also, using (25) and (26), we get $Q_2^2=\frac{\left(h_2(y)\right)^2(\mu+1)^2\upsilon_1^2}{4\left(\eta+i\rho\right)^2}$. Thus, from (29), we have

$$Q_3-\rho Q_2^2=\frac{h_2(y)\left(\upsilon_2-t_2\right)(2\mu+1)}{3(\eta+i\rho)}+(1-\rho)F_2^2$$

$$=\frac{h_2(y)\left(\upsilon_2-t_2\right)(2\mu+1)}{3(\eta+i\rho)}+(1-\rho)\frac{\left(h_2(y)\right)^2(\mu+1)^2\upsilon_1^2}{4\left(\eta+i\rho\right)^2}.$$

From Lemma 2.4, we get $2\upsilon_2 = \upsilon_1^2 + \alpha(4-\upsilon_1^2)$ and $2t_2 = t_1^2 + \zeta(4-t_1^2)$, $|\alpha| \le 1$, $|\zeta| \le 1$, and using (25), we have

$$\upsilon_2 - t_2 = \frac{4-\upsilon_1^2}{2}(\alpha - \zeta),$$

and thus

$$Q_3 - \rho Q_2^2 = \frac{h_2(y)(4-\upsilon_1^2)(2\mu+1)(\alpha-\zeta)}{6(\eta+i\rho)} + (1-\rho)\frac{(h_2(y))^2(\mu+1)^2\upsilon_1^2}{4(\eta+i\rho)^2}.$$

Using the triangle inequality, assuming that $\upsilon_1 = j \in 0,2]$ and taking $|\alpha| = m$, $|\zeta| = r$, $m,r \in 0,1]$, thus, we get

$$\left|Q_3 - \rho Q_2^2\right| \le \frac{|h_2(y)|(2\mu+1)(4-j^2)(m+r)}{6|\eta+i\rho|} + |1-\rho|\frac{(h_2(y))^2(\mu+1)^2 j^2}{4|\eta+i\rho|^2}. \tag{32}$$

Using (2) for equation (32), we have

$$\left|Q_3 - \rho Q_2^2\right| \le \frac{|\varepsilon y|(2\mu+1)(4-j^2)(m+r)}{6|\eta+i\rho|} + |1-\rho|\frac{\varepsilon^2 y^2(\mu+1)^2 j^2}{4|\eta+i\rho|^2}.$$

Assume that: $K_1(j) = \dfrac{\varepsilon^2 y^2(\mu+1)^2 j^2 |1-\rho|}{4|\eta+i\rho|^2} \ge 0$ and $K_2(j) = \dfrac{|\varepsilon y|(2\mu+1)(4-j^2)}{6|\eta+i\rho|^2} \ge 0$, we can rewritten the inequality (32) as:

$$\left|Q_3 - \rho Q_2^2\right| \le K_1(j) + K_2(j)(m+r) =: B(m,r), m,r \in 0,1].$$

Therefore,

$$\max\{B(m,r) : m,r \in 0,1]\} = B(1,1) = K_1(j) + 2K_2(j) =: M(j), j \in 0,2]$$

where

$$M(j) = \frac{\varepsilon^2 y^2(\mu+1)^2}{4|\eta+i\rho|^2}\left(|1-\rho| - \frac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2}\right)j^2 + \frac{4|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|}.$$

Since

$$M'(j) = \frac{\varepsilon^2 y^2(\mu+1)^2}{2|\eta+i\rho|^2}\left(|1-\rho| - \frac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2}\right)j,$$

it is clear that $M'(j) \le 0$ iff $|1-\rho| \le \dfrac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2}$. Hence, the function M is a decreasing on $[0,2]$; therefore,

$$\max\{M(j) : j \in 0,2]\} = M(0) = \frac{4|\varepsilon y|(2\mu+1)}{3|\eta+i\rho|}.$$

Also, $M'(j) \geq 0$ iff $|1-\rho| \geq \frac{4(2\mu+1)|\eta+i\rho|}{3|\varepsilon y|(\mu+1)^2}$. So, M is an increasing function over $[0,2]$, so

$$\max\{M(j): j \in 0,2]\} = M(2) = \frac{\varepsilon^2 y^2(\mu+1)^2|1-\rho|}{|\eta+i\rho|^2}$$

and the estimation (17) has been confirmed to be accurate.

Theorem 2.6 *Let* $F \in BU$ *given by (8). If* $F \in BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$. *Then*

$$|Q_2| \leq \min\left\{\frac{|\varepsilon y|(\mu+1)}{|\eta+i\rho|}, \frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\varepsilon^2 y^2\left(\frac{\eta+i\rho}{2\mu+1}+\frac{(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{2\left(\gamma\varepsilon y^2+\alpha\delta\right)(\eta+i\rho)^2}{(\mu+1)^2}\right|}}\right\},$$

$$|Q_3| \leq \min\left\{\frac{\varepsilon^2 y^2(\mu+1)^2}{|\eta+i\rho|^2}+\frac{2|\varepsilon y|(2\mu+1)}{|\eta+i\rho|},\right.$$

$$\left.\frac{2|\varepsilon y|^3}{\left|\varepsilon^2 y^2\left(\frac{\eta+i\rho}{2\mu+1}+\frac{(\eta+i\rho)(\eta+i\rho-1)}{(\mu+1)^2}\right)-\frac{2\left(\gamma\varepsilon y^2+\alpha\delta\right)(\eta+i\rho)^2}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|(2\mu+1)}{|\eta+i\rho|}\right\}$$

and

$$\left|Q_3-\rho Q_2^2\right| \leq \begin{cases} \frac{4|\varepsilon y|(2\mu+1)}{|\eta+i\rho|} & |1-\rho| < \frac{(2\mu+1)|\eta+i\rho|}{|\varepsilon y|(\mu+1)^2}, \\ \frac{4\varepsilon^2 y^2(\mu+1)^2|1-\rho|}{|\eta+i\rho|^2} & |1-\rho| \geq \frac{(2\mu+1)|\eta+i\rho|}{|\varepsilon y|(\mu+1)^2}. \end{cases}$$

Proof. Let $F \in BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$. So from (14) and (15), we can write

$$\left[\frac{EF_\mu^i(\varsigma)}{\varsigma}\right]^{\eta+i\rho} = GH(y,T(\varsigma))+1-\alpha, \varsigma \in \Delta$$

and

$$\left[\frac{EG_\mu^i(w)}{w}\right]^{\eta+i\rho} = GH(y,V(w))+1-\alpha,\ w \in \Delta.$$

Thus we have

$$\left[\frac{EF_{\mu}^{i}(\varsigma)}{\varsigma}\right]^{\eta+i\rho}=1+\frac{\upsilon_1}{4}\varsigma+\frac{1}{48}\left(12\upsilon_2-7\upsilon_1^2\right)\varsigma^2+\frac{1}{192}\left(17\upsilon_1^3-56\upsilon_1\upsilon_2+48\upsilon_3\right)\varsigma^3+\cdots,\varsigma\in\Delta. \tag{33}$$

and

$$\left[\frac{EG_{\mu}^{i}(w)}{w}\right]^{\eta+i\rho}=1+\frac{t_1}{4}w+\frac{1}{48}\left(12t_2-7t_1^2\right)w^2+\frac{1}{192}\left(17t_1^3-56t_1t_2+48t_3\right)w^3+\cdots,w\in\Delta. \tag{34}$$

From the equations (33) and (34), we get

$$\frac{\eta+i\rho}{\mu+1}Q_2=h_2(y)\upsilon_1, \tag{35}$$

$$\frac{(\eta+i\rho)(\eta+i\rho-1)}{2(\mu+1)^2}Q_2^2+\frac{\eta+i\rho}{2(2\mu+1)}Q_3=h_2(y)\upsilon_2+h_3(y)\upsilon_1^2, \tag{36}$$

$$-\frac{\eta+i\rho}{\mu+1}Q_2=h_2(y)t_1, \tag{37}$$

and

$$\left[\frac{\eta+i\rho}{2\mu+1}+\frac{(\eta+i\rho)(\eta+i\rho-1)}{2(\mu+1)^2}\right]Q_2^2-\frac{\eta+i\rho}{2(2\mu+1)}Q_3=h_2(y)t_2+h_3(y)t_1^2. \tag{38}$$

We obtain the results of Theorem 2.6 by using the final four equations and the same method used to prove Theorem 2.5.

3. Corollaries and Remark

For the subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$, numerous corollaries can be obtained for specific values of η,ρ in Theorems 2.5 and 2.6, for example.

Corollary 3.1 *If* $F\in AEH(\alpha,\varepsilon,\gamma,\delta,1,\rho)$, *then*

$$|Q_2|\le\min\left\{\frac{|\varepsilon y|(\mu+1)}{2|1+i\rho|},\frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\varepsilon^2y^2\left(\frac{3(1+i\rho)}{2\mu+1}+\frac{4i\rho(1+i\rho)}{(\mu+1)^2}\right)-\frac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)(1+i\rho)^2}{(\mu+1)^2}\right|}}\right\},$$

$$|Q_3|\le\min\left\{\frac{\varepsilon^2y^2(\mu+1)^2}{4|1+i\rho|^2}+\frac{2|\varepsilon y|(2\mu+1)}{3|1+i\rho|},\right.$$

$$\left.\frac{2|\varepsilon y|^3}{\left|\varepsilon^2y^2\left(\frac{3(1+i\rho)}{2\mu+1}+\frac{4i\rho(1+i\rho)}{(\mu+1)^2}\right)-\frac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)(1+i\rho)^2}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|(2\mu+1)}{3|1+i\rho|}\right\}$$

and

$$\left|Q_3-\rho Q_2^2\right|\le\begin{cases}\dfrac{4|\varepsilon y|(2\mu+1)}{3|1+i\rho|} & |1-\rho|<\dfrac{4(2\mu+1)|1+i\rho|}{3|\varepsilon y|(\mu+1)^2},\\[2ex] \dfrac{\varepsilon^2y^2(\mu+1)^2|1-\rho|}{|1+i\rho|^2} & |1-\rho|\ge\dfrac{4(2\mu+1)|1+i\rho|}{3|\varepsilon y|(\mu+1)^2}.\end{cases}$$

Corollary 3.2 *If* $F\in AEH(\alpha,\varepsilon,\gamma,\delta,\eta,0)$, *then*

$$|Q_2|\le\min\left\{\frac{|\varepsilon y|(\mu+1)}{2\eta},\frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\varepsilon^2y^2\left(\dfrac{3\eta}{2\mu+1}+\dfrac{4\eta(\eta-1)}{(\mu+1)^2}\right)-\dfrac{8\eta^2\left(\gamma\varepsilon y^2+\alpha\delta\right)}{(\mu+1)^2}\right|}}\right\},$$

$$|Q_3|\le\min\left\{\frac{\varepsilon^2y^2(\mu+1)^2}{4\eta^2}+\frac{2|\varepsilon y|(2\mu+1)}{3\eta},\right.$$

$$\left.\frac{2|\varepsilon y|^3}{\left|\varepsilon^2y^2\left(\dfrac{3\eta}{2\mu+1}+\dfrac{4\eta(\eta-1)}{(\mu+1)^2}\right)-\dfrac{8\eta^2\left(\gamma\varepsilon y^2+\alpha\delta\right)}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|(2\mu+1)}{3\eta}\right\}$$

and

$$\left|Q_3-\rho Q_2^2\right|\le\begin{cases}\dfrac{4|\varepsilon y|(2\mu+1)}{3\eta} & |1-\rho|<\dfrac{4\eta(2\mu+1)}{3|\varepsilon y|(\mu+1)^2},\\[2ex] \dfrac{\varepsilon^2y^2(\mu+1)^2|1-\rho|}{\eta^2} & |1-\rho|\ge\dfrac{4\eta(2\mu+1)}{3|\varepsilon y|(\mu+1)^2}.\end{cases}$$

Corollary 3.3 *If* $F\in AEH(\alpha,\varepsilon,\gamma,\delta,1,0)$, *then*

$$|Q_2|\le\min\left\{\frac{|\varepsilon y|(\mu+1)}{2},\frac{|\varepsilon y|\sqrt{2|\varepsilon y|}}{\sqrt{\left|\dfrac{3\varepsilon^2y^2}{2\mu+1}-\dfrac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)}{(\mu+1)^2}\right|}}\right\},$$

$$|Q_3|\le\min\left\{\frac{\varepsilon^2y^2(\mu+1)^2}{4}+\frac{2|\varepsilon y|(2\mu+1)}{3},\right.$$

$$\left.\frac{2|\varepsilon y|^3}{\left|\frac{3\varepsilon^2 y^2}{2\mu+1}-\frac{8\left(\gamma\varepsilon y^2+\alpha\delta\right)}{(\mu+1)^2}\right|}+\frac{2|\varepsilon y|\left(2\mu+1\right)}{3}\right\}$$

and

$$\left|Q_3-\rho Q_2^2\right|\leq\begin{cases}\frac{4}{3}|\varepsilon y|\left(2\mu+1\right) & |1-\rho|<\frac{4\left(2\mu+1\right)}{3|\varepsilon y|(\mu+1)^2},\\ \varepsilon^2 y^2(\mu+1)^2\,|1-\rho| & |1-\rho|\geq\frac{4\left(2\mu+1\right)}{3|\varepsilon y|(\mu+1)^2}.\end{cases}$$

Remark 3.4 *For the subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and $BEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$, we can derive numerous corollaries for specific values of $\alpha,\varepsilon,\gamma,\delta,\eta,\rho$ in Theorems 2.5 and 2.6. In particular, in view of Remark 1.1, we can derive several results related to Lucas polynomials, Fibonacci polynomials, Pell polynomials and Pell-Lucas polynomials, Chebyshev polynomials of the first kind and second kind.*

4. Conclusions

In this paper, we introduced the inclusive subfamilies $AEH(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ and BEH $(\alpha,\varepsilon,\gamma,\delta,\eta,\rho)$ of complex order using the Error functions subordinate to Horadam polynomials. For functions in these subfamilies, we derive the estimations of the initial coefficients $|Q_2|$ and $|Q_3|$, and the functional $|Q_3-\rho Q_2^2|$.

The results of this study provide opportunities for more research, especially because of the original characterizations and evidence offered. These findings open the door for further study of more special functions within the analytic and bi-univalent function subfamilies of complex order, in addition to enhancing the theory of these subfamilies. The interaction of the introduced subfamilies, Horadam polynomials, and the error function may lead to new avenues for the investigation of complex functions and their uses.

References

[1] H. Alzer, Error functions inequalities, *Adv. Comput. Math.*, 33, (2010), 349–379.
[2] A. Amourah, B.A. Frasin, G. Murugusundaramoorthy and T. Al-Hawary, Bi-Bazilevic functions of order ε $+iL$ associated with (p; q)-Lucas polynomials, *AIMS Mathematics*, 6(5), (2021), 4296–4305.
[3] A. Amourah, B. A. Frasin and T. M. Seoudy, An application of Miller-Ross-type Poisson distribution on certain subclasses of bi-univalent functions subordinate to Gegenbauer polynomials, *Mathematics*, 10, (2022), 2462.
[4] A. Amourah, B.A. Frasin and T. Al-Hawary, Coefficient estimates for a subclass of bi-univalent functions associated with symmetric q-derivative operator by means of the Gegenbauer polynomials. *Kyungpook Mathematical Journal*, 62(2), (2022), 257–269.
[5] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical functions with formulas, Graphs and Matematical Tables*, Dover Publications Inc. New York, (1972).
[6] M.P. Chen, *On functions satisfying* $Re\left\{\frac{f(z)}{z}\right\}>\alpha$, Tamkang Journal of Mathematics, 5, (1974), 231–234.
[7] D. Coman, The radius of starlikeness for error function, *Stud. Univ. Babes Bolyal Math.*, 36, (1991), 13–16.
[8] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, (1983).
[9] A. Elbert and A. Laforgia, The zeros of the complementary error function, *Numer. Algorithms*, 49, (2008), 153–157.
[10] T.G. Ezrohi, *Certain estimates in special classes of univalent functions in the unitcircle*, Doporidi Akademii Nauk Ukrains, RSR 2(1965), 984–988.

[11] T. Al-Hawary, B.A. Frasin, D. Breaz and L-I. Cotrla, Inclusive Subclasses of Bi-Univalent Functions Defined by Error Functions Subordinate to Horadam Polynomials, *Symmetry*, 17, 2025, 211.
[12] T. Al-Hawary, A. Amourah and B.A. Frasin, Fekete-Szegö inequality for bi-univalent functions by means of Horadam polynomials, *Boletin de la Sociedad Matematica Mexicana*, 27(3), (2021), 1–12.
[13] T. Al-Hawary, B.A. Frasin and J. Salah, Comprehensive Subfamilies of Bi-Univalent Functions Defined by Error Function Subordinate to Euler Polynomials, *Symmetry*, 17, (2025), 256.
[14] A. Hussen, An application of the Mittag-Leffler-type Borel distribution and Gegenbauer polynomials on a certain subclass of bi-univalent functions, *Heliyon*, 10, (2024), e31469.
[15] A.F. Horadam, Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quart.*, 3, (1965), 161–176.
[16] A.F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J.*, 32, (1965), 437–446.
[17] M. Illafe, M.H. Mohd and F. Yousef, Supramaniam, S. A Subclass of Bi-univalent Functions Defined by aSymmetric q-Derivative Operator and Gegenbauer Polynomials, *European J. Pure App. Math.*, 17, (2024), 2467–2480.
[18] H. Irmak, Various results for series expansions of the error functions with the complex variable and some of their implications, *Turkish Journal of Mathematics,* 44, (2020), 1640–1648.
[19] H. Irmak and M. Duran, Some results for error function with complex argument, *Journal of Orissa Mathematical Society*, 38(1-2), (2019), 53–68.
[20] R.J. Libera and E.J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P, *Proc. Am. Math. Soc.*, 87, (1983), 251–257.
[21] S. Miller and P. Mocanu, *Differential Subordination: theory and applications*, CRC Press, New York, (2000).
[22] N.H. Mohammed, N.E. Cho, E.A. Adegani and T. Bulboaca, Geometric properties of normalized imaginary error function, *Stud. Univ. Babe s-Bolyai Math.,* 67(2), (2022), 455–462.
[23] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, *Abst. App. Anal.*, 2013, (2013), Article ID 573017, 3 pages.
[24] C. Ramachandran, L. Vanitha and S. Kanas, Certain results on q-starlike and q-convex error functions, *Math. Slovaca.*, 68, (2018), 361–368.
[25] Z. Peng, G. Murugusundaramoorthy and T. Janani, Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator, *J. Comp. Anal.*, 2014, (2014), Article ID 693908, 6 pages.
[26] F.S.M. Al Sarari, B.A. Frasin, T. Al-Hawary and S. Latha, A few results on generalized Janowski type functions associated with (j, k)-symmetrical functions, *Acta Univ. Sapientiae Math.*, 8(2), (2016), 195–205.
[27] H.M. Srivastava, A.K.Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *App. Math. Lett.*, 23(10), (2010), 1188–1192.
[28] F. Yousef, F.; B.A. Frasin and T. Al-Hawary, Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials, *Filomat*, 32(9), (2018), 3229–3236.
[29] F. Yousef, S. Alroud and M. Illafe, A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind, *Boletin de la Sociedad Matemática Mexicana,* 26, (2020), 329–339.
[30] P. Zaprawa, *Estimates of initial coefficients for bi-univalent functions,* Abstr. Appl. Anal. 2014, 357480. held at the University of Durham, Durham, UK, 1–20 July 1979; Academic Press: New York, NY, USA; London, UK, (1980).