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Fully implicit differences method for solving couple parabolic system with variable coefficients

Jamil A. Al-Hawasy[a]*, Marwa A. Jawad[b], Doaa K. Jasim[c], Lamyaa H. Ali[d]

[a,d]Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq; [b]Department of Mathematics, College of Basic Education, University of Diyala, Iraq; [c]Technical college of management, Middle Technical University of Baghdad, Baghdad, Iraq

Abstract

This article concerns with introducing new technique to solve a new type of PDE described by coupled parabolic system with variable coefficients (CPSVC) by utilizing the fully finite implicit differences method (FFIDM). At each discrete value of time t_j the proposed technique is used to transform the CPSVC into a couple linear algebraic system (CLS) that they are solved using the Gauss elimination method (GEM) to get the numerical couple solution (NCS) for the problem. The consistency of the method is studied so as the stability. Some examples are given and the results are described by tables and figures to illustrate the accuracy for the proposed technique, it is concluded that this method is accurate and suitable for solving such systems.

Mathematics Subject Classification (2020): 35Cxx

Key words and Phrases: Coupled linear Parabolic System, Consistency Fully implicit finite difference method, Stability, Variable Coefficients.

1. Introduction

The finite difference method (FDM) is a popular and powerful numerical technique for solving PDEs including parabolic equations that usually describes the diffusion processes, such as heat conduction, fluid flow, and option pricing. In the implicit form, the method approximates the solution at a new future time step based on the values at the current and future time steps. This approach often leads to unconditionally stable schemes, making it suitable for a wide range of problems.

Email addresses: jhawassy17@uomustansiriyah.edu.iq (Jamil A. Ali Al-Hawasy); basicmathte1@uodiyala.edu.iq (Marwa A. Jawad); D0aa.katib@mtu.edu.iq (Doaa K. Jasim); lamya_h2@uomustansiriyah.edu.iq (Lamyaa H. Ali)

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Many researchers during all the time are interested to study in general the numerical solutions of the PDEs, according to this interest, many investigators and researcher found and utilized numerous type of techniques and methods to solve PDEs. Such as; the spectral method and the finite difference method (FDM) for solving elliptic PDE [1] & [2], the FDM [3], the Galerkin finite element method (GFE) [4], the mixed Homotopy perturbation method with Crank-Nicolson method (CNM)[5], A hybrid numerical method (Chebyshev functions and collocation method) [6–8] and mixed GFM with the FDM used to solve different types of parabolic PDE [9], while interpolating wavelets [10], the FDM [11] and the mixed GFE with CNM used to solve different type of hyperbolic PDEs [12].

Later the researchers extended their concerns about studying more general type of PDEs as [13] who they used the GFE to solve couple elliptic, while [14] interested about using the mixed GFM with the FFIDM to solve couple system of PDE with constant coefficients.

All these investigators, motivate us to aim about studding the solution for CPSVC (which represents an extension to the model of problem in [14] form constants coefficients into variable coefficients) by utilizing the FFIDM, besides the study of approximate solution for couple of PDEs play an important rule to solve many complicated problems like as the optimal control problems in [15–17].

This paper starts with the description of the CPSVC, the NCS is obtained from the discretization of the continuous CPSVC, by using the FFIDM. This technique is transformed the CPSVC into a CLS at any discrete time t_j, we used the GEM to solve this CLS. Also, the consistency (the stability) of this method is proved (is ensured). Finally, some examples are given to find the NCS utilizing the FFIDM, the results are described by tables and figures to show the efficiency and the accuracy of the method.

2. Description of the CPSVC}

Let $\Omega=\left\{\vec{y}=\left(y_1,y_2\right)\in R^2:\ 0<y_1,y_2<1\right\}\subset\mathbb{R}^2$, be the region with boundary $\partial\Omega$, let $I=[0,T], 0<T<\infty$, and Q = I x Ω ,then the CPSVC are represented as:

$$U_{1t}-\sum_{r=1}^{2}\frac{\partial}{\partial y_r}\left[a_{1r}\left(\vec{y},t\right)\frac{U_1}{y_r}\right];c_1\left(\vec{y},t\right)U_1-d\left(\vec{y},t\right)U_2=F_1\left(\vec{y},t\right),\ in\ Q \tag{1}$$

$$U_{2t}-\sum_{r=1}^{2}\frac{\partial}{\partial y_r}\left[a_{2r}\left(\vec{y},t\right)\frac{?U_2}{?\,y_r}\right]+c_2\left(\vec{y},t\right)U_2+d\left(\vec{y},t\right)U_1=F_2\left(\vec{y},t\right),\ in\ Q \tag{2}$$

with the initial conditions

$$U_i\left(\vec{y},0\right)=U_i^0(\vec{y}),\ \text{on}\ \Omega\times I,(i=1,2) \tag{3}$$

and the boundary conditions

$$U_i\left(\vec{y},t\right)=0,\ \text{on}\ \Omega\times I,(i=1,2) \tag{4}$$

where $U_1=U_1\left(\vec{y},t\right),\ U_2=U_2\left(\vec{y},t\right)\in C^2\left(Q\right),\ U_{1t}=\frac{\partial U_1}{\partial t},\ U_{2t}=\frac{\partial U_2}{\partial t},\ F_1\left(\vec{y},t\right)\ \&\ F_2\left(\vec{y},t\right)\in C\left(Q\right)$ and are given. The “classical solution” of ((1)-(4)) is $\vec{U}\left(\vec{y},t\right)=\left(U_1\left(\vec{y},t\right),U_2\left(\vec{y},t\right)\right)\in(C^2\left(Q\right))^2$, for all $\vec{y}$ in Ω.

3. The Numerical Couple Solution

To find the NCS $\vec{U}^n=(U_1^n,U_2^n)$ of ((1)-(4)), using the FFIDM, the domain of the problem is discretized to the points $(y_{i1},y_{j2}), i,j=1,2,\ldots,N$, (where $\Delta y_1=\Delta y_2=h=1/N$, and $y_{il}=ih, l=1,2$) for the space variable and $t_k = k\Delta t,\ k = 0,1,2,\ldots, NT-1(\Delta t = T/NT)$ for the time. The implicit difference formula for $\frac{\partial U_p}{\partial t}$ and the central difference formulas for $\frac{\partial U_p\left(t_k y_{i1}, y_{j2}\right)}{\partial y_l}$, and $\frac{\partial^2 U_p\left(t_{k,} y_{i1}, y_{j2}\right)}{\partial y_l^2}$ (for $i,j = 1,2,\ldots,N,$ and $p = 1,2$):

$$\frac{\partial U_p}{\partial t}\left(t_k y_{i1}, y_{j2}\right) = \frac{U_p^{k+1}(i,j) - U_p^k(i,j)}{\Delta t} + O(\Delta t), \frac{\partial U_p\left(t_k x_{i1}, x_{j2}\right)}{\partial y_1} = \frac{U_p^{k+1}(i+1,j) - U_p^{k+1}(i-1,j)}{2h} + O(h^2),$$

$$\frac{\partial U_p\left(t_k x_{i1}, x_{j2}\right)}{\partial y_2} = \frac{U_p^{k+1}(i,j+1) - U_p^{k+1}(i,j-1)}{2\Delta y_2} + O(h^2),$$

$$\frac{\partial^2 U_p\left(t_k x_{i1}, x_{j2}\right)}{\partial y_1^2} = \frac{U_p^{k+1}(i+1,j) - 2U_p^{k+1}(i,j) + U_p^{k+1}(i-1,j)}{h^2} + O(h^2), \text{ and}$$

$$\frac{\partial^2 U_p\left(t_k x_{i1}, x_{j2}\right)}{\partial y_2^2} = \frac{U_p^{k+1}(i,j+1) - 2U_p^{k+1}(i,j) + U_p^{k+1}(i,j-1)}{h^2} + O(h^2).$$

Hence equations ((1)-(4)) with using the above difference formulas, become

$$\frac{U_1^{k+1}(i,j) - U_1^k(i,j)}{\Delta t} - a_{11}\left(\vec{y}_{ij}\right)\left[\frac{U_1^{k+1}(i+1,j) - 2U_1^{k+1}(i,j) + U_1^{k+1}(i-1,j)}{h^2}\right] - a_{11_{y_1}}\left(\vec{y}_{ij}\right)\left[\frac{U_1^{k+1}(i+1,j) - U_1^{k+1}(i-1,j)}{2h}\right]$$

$$-a_{12}\left(\vec{y}_{ij}\right)\left[\frac{U_1^{k+1}(i,j+1) - 2U_1^{k+1}(i,j) + U_1^{k+1}(i,j-1)}{h^2}\right] - a_{12_{y_2}}\left(\vec{y}_{ij}\right)\left[\frac{U_1^{k+1}(i,j+1) - U_1^{k+1}(i,j-1)}{2h}\right]$$

$$+c_1\left(\vec{y}_{ij},t\right)U_1^{k+1}(i,j) - d\left(\vec{y}_{ij},t\right)U_2^{k+1}(i,j) = F_1^k(i,j) \tag{5}$$

$$\frac{U_2^{k+1}(i,j) - U_2^k(i,j)}{\Delta t} - a_{21}\left(\vec{y}_{ij}\right)\frac{U_2^{k+1}(i+1,j) - 2U_2^{k+1}(i,j) + U_2^{k+1}(i-1,j)}{h^2}$$

$$-a_{21_{y_1}}\left(\vec{y}_{ij}\right)\left[\frac{U_2^{k+1}(i+1,j) - U_2^{k+1}(i-1,j)}{2h}\right] - a_{22}\left(\vec{y}_{ij}\right)\left[\frac{U_2^{k+1}(i,j+1) - 2U_2^{k+1}(i,j) + U_2^{k+1}(i,j-1)}{h^2}\right]$$

$$+c_2\left(\vec{y}_{ij},t\right)U_2^{k+1}(i,j) + d\left(\vec{y}_{ij},t\right)U_1^{k+1}(i,j) = F_2^k(i,j) \tag{6}$$

where $U_p^{k+1}\left(t_k, y_{i,1}, y_{j2}\right) = U_p^k(i,j),\ U_p^{k+1}\left(t_k, y_{i\pm1,1}, y_{j2}\right) = U_p^k(i\pm1,j),\quad U_p^{k+1}\left(t_k, y_{i1}, y_{j\pm1,2}\right) = U_p^k(i,j\pm1),$ for $i,j = 1,2,\ldots,N,\ k = 0,1,2,\ldots,NT-1,\ p = 1,2$ and $l = 1,2$.

Let $r = \frac{\Delta t}{h^2}, s = \frac{\Delta t}{2h},\ r_1 = 1 + 2ra_{11}\left(\vec{y}_{ij}\right) + 2ra_{12}\left(\vec{y}_{ij}\right) + \Delta t c_1\left(\vec{y}_{ij}\right),\quad r_2 = ra_{11}\left(\vec{y}_{ij}\right) + sa_{11_{y_1}}\left(\vec{y}_{ij}\right),\quad r_3 = ra_{11}\left(\vec{y}_{ij}\right) - sa_{11_{y_1}}\left(\vec{y}_{ij}\right),\ r_4 = ra_{12}\left(\vec{y}_{ij}\right) + sa_{12_{y_2}}\left(\vec{y}_{ij}\right),\ r_5 = ra_{12}\left(\vec{y}_{ij}\right) - sa_{12_{y_2}}\left(\vec{y}_{ij}\right)$

$s_1 = 1 + 2ra_{21}\left(\vec{y}_{ij}\right) + 2ra_{22}\left(\vec{y}_{ij}\right) + \Delta t c_2\left(\vec{y}_{ij}\right), s_2 = ra_{21}\left(\vec{y}_{ij}\right) + sa_{21_{y_1}}\left(\vec{y}_{ij}\right),$

$s_3 = ra_{21}\left(\vec{y}_{ij}\right) - sa_{21_{y_1}}\left(\vec{y}_{ij}\right), s_4 = ra_{22}\left(\vec{y}_{ij}\right) + sa_{22_{y_2}}\left(\vec{y}_{ij}\right),$ and $s_5 = ra_{22}\left(\vec{y}_{ij}\right) - sa_{22_{y_2}}\left(\vec{y}_{ij}\right)$.

Using these equalities in ((5)-(6)), to get

$$r_1U_1^{k+1}(i,j) - r_2U_1^{k+1}(i+1,j) - r_3U_1^{k+1}(i-1,j) - r_4U_1^{k+1}(i,j+1) - r_5U_1^{k+1}(i,j-1) -$$

$$-\Delta t d\left(\vec{y}_{ij},t\right)U_2^{k+1}(i,j) = U_1^k(i,j) + \Delta t\, F_1^k(i,j) \tag{7}$$

and

$$s_1 U_2^{k+1}(i,j) - s_2 U_2^{k+1}(i+1,j) - s_3 U_2^{k+1}(i-1,j) - s_4 U_2^{k+1}(i,j+1) - s_5 U_2^{k+1}(i,j-1) -$$

$$-\Delta t d(\vec{y},t) U_1^{k+1}(i,j) = U_2^k(i,j) + \Delta t\, F_2^k(i,j) \tag{8}$$

Equation ((7)-(8)) represent the FFIDM formula for the CPSVC and can be expressed as the following CLS: $A^{k+1}\vec{U} = (\vec{U})^k + \vec{b}$

when $\vec{U} = (U_1, U_2)$, the matrix A^k is consisted of three matrices $A_1^k = (a_{ij}^k)_{2N\times 2N}$, $A_2^k = (a_{ij}^{k*})_{2N\times 2N}$ and $A_3 = (\bar{a}_{ij}^k)_{2N\times 2N}$, s.t.

$$a_{ij}^k = \begin{cases} r_1 & i=j=1,2,\ldots,2N \\ -r_4 & i=1,2,\ldots,2N-1, j=i+1, i\neq (N)l, l=1,2,3 \\ -r_2 & i=1,2,3,\ldots,N(N-1)\ ,\ j=N+i, \\ -r_5 & j=1,2,\ldots,2N-1\ , i=j+1, j\neq (N)l, l=1,2,3 \\ -r_3 & j=1,2,3,\ldots,N(N-1)\ , i=N+j, \\ 0 & o.w \end{cases} \tag{9}$$

$$a_{ij}^{k*} = \begin{cases} s_1 & i=j=1,2,\ldots,2N \\ -s_4 & i=1,2,\ldots,2N-1, j=i+1, i\neq (N)l, l=1,2,3 \\ -s_2 & i=1,2,3,\ldots,N(N-1)\ ,\ j=N+i, \\ -s_5 & j=1,2,\ldots,2N-1\ , j\neq (N)l, i=j+1, l=1,2,3 \\ -s_3 & j=1,2,3,\ldots,N(N-1)\ , i=N+j \\ 0 & o.w \end{cases},\quad \bar{a}_{ij}^k = \begin{cases} \Delta t & i=j=1,2,\ldots,2N \\ 0 & o.w \end{cases} \tag{10}$$

and, $\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}_{2N\times 1}$, with $\vec{b}_l = (b_{lij}^k)_{N\times 1} = U_l^k(i,j) + \Delta t\, F_l^k(i,j)$, for $i,j=1,2,\ldots,M$, $l=1,2$.

4. Consistency and Stability of the CPSVC

From the Taylor series, each of the term in with p = 1,2 ((7)-(8)) is expressed as:

$$U_p^{k+1}(i,j) = U_p^k(i,j) + (\Delta t)\frac{\partial U_p^k(i,j)}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 U_p^k(i,j)}{\partial t^2} + O(\Delta t^3), \tag{11}$$

$$U_p^{k+1}(i\pm 1,j) = U_p^k(i,j) \pm h\frac{\partial U_p^k(i,j)}{\partial y_p} + \frac{h^2}{2}\frac{\partial^2 U_p^k(i,j)}{\partial y_p^2} \pm \frac{h^3}{6}\frac{\partial^3 U_p^k(i,j)}{\partial y_p^3} + \frac{h^4}{24}\frac{\partial^4 U_p^k(i,j)}{\partial y_p^4} + O(h^5) \tag{12}$$

$$U_p^{k+1}(i,j\pm 1) = U_p^k(i,j) \pm h\frac{\partial U_p^k(i,j)}{\partial y_p} + \frac{h^2}{2}\frac{\partial^2 U_p^k(i,j)}{\partial y_p^2} \pm \frac{h^3}{6}\frac{\partial^3 U_p^k(i,j)}{\partial y_p^3} + \frac{h^4}{24}\frac{\partial^4 U_p^k(i,j)}{\partial y_p^4} + O(h^5) \tag{13}$$

From ((11)-(13)), one gets that

$$\frac{\partial U_1^k(i,j)}{\partial t} - a_{11}(\vec{y}_{ij})\left[\frac{U_1^{k+1}(i+1,j) - 2U_1^{k+1}(i,j) + U_1^{k+1}(i-1,j)}{h^2}\right] - a_{11_{y_1}}(\vec{y}_{ij})\left[\frac{U_1^{k+1}(i+1,j) - U_1^{k+1}(i-1,j)}{2h}\right]$$

$$-a_{12}(\vec{y}_{ij})\left[\frac{U_1^{k+1}(i,j+1) - 2U_1^{k+1}(i,j) + U_1^{k+1}(i,j-1)}{h^2}\right] - \grave{a}_{12_{y_2}}(\vec{y}_{ij})\left[\frac{U_1^{k+1}(i,j+1) - U_1^{k+1}(i,j-1)}{2h}\right]$$

$$+c_1(\vec{y}_{ij},t)U_1^{k+1}(i,j)-d(\vec{y}_{ij},t)U_2^{k+1}(i,j)-F_1^k(i,j)=$$

$$\frac{\Delta t}{2}\frac{\partial^2 U_1^k(i,j)}{\partial t^2}-O(\Delta t^2)-a_{11}(\vec{y}_{ij})h^2\frac{\partial^4 U_1^k(i,j)}{\partial y_1^4}-O(h^4)-\grave{a}_{11y_1}(\vec{y}_{ij})\frac{h^2}{6}\frac{\partial^3 U_1^k(i,j)}{\partial y_1^3}-O(h^4)$$

$$-a_{12}(\vec{y}_{ij})h^2\frac{\partial^4 U_1^k(i,j)}{\partial y_2^4}-O(h^4)-\grave{a}_{12y_2}(\vec{y}_{ij})\frac{h^2}{6}\frac{\partial^3 U_1^k(i,j)}{\partial y_2^3}-O(h^4)$$

And

$$\frac{\partial U_2^k(i,j)}{\partial t}-a_{21}(\vec{y}_{ij})\left[\frac{U_2^{k+1}(i+1,j)-2U_2^{k+1}(i,j)+U_2^{k+1}(i-1,j)}{h^2}\right]-\grave{a}_{21y_1}(\vec{y}_{ij})\left[\frac{U_2^{k+1}(i+1,j)-U_2^{k+1}(i-1,j)}{2h}\right]$$

$$-a_{22}(\vec{y}_{ij})\left[\frac{U_2^{k+1}(i,j+1)-2U_2^{k+1}(i,j)+U_2^{k+1}(i,j-1)}{h^2}\right]-\grave{a}_{21y_2}(\vec{y}_{ij})\left[\frac{U_2^{k+1}(i,j+1)-U_2^{k+1}(i,j-1)}{2h}\right]$$

$$+c_2(\vec{y}_{ij},t)U_2^{k+1}(i,j)-d(\vec{y}_{ij},t)U_1^{k+1}(i,j)-F_2^k(i,j)=$$

$$\frac{\Delta t}{2}\frac{\partial^2 U_2^k(i,j)}{\partial t^2}-O(\Delta t^2)-a_{21}(\vec{y}_{ij})h^2\frac{\partial^4 U_2^k(i,j)}{\partial y_1^4}-O(h^4)-\grave{a}_{21y_1}(\vec{y}_{ij})\frac{h^2}{6}\frac{\partial^3 U_2^k(i,j)}{\partial y_1^3}-O(h^4)$$

$$-a_{22}(\vec{y}_{ij})h^2\frac{\partial^4 U_2^k(i,j)}{\partial y_2^4}-O(h^4)-\grave{a}_{21y_2}(\vec{y}_{ij})\frac{h^2}{6}\frac{\partial^3 U_2^k(i,j)}{\partial y_2^3}-O(h^4)$$

As Δt and h tend to zero, the above two equality give

$$U_{1t}^k(i,j)-a_{11}(\vec{y}_{ij})\frac{\partial^2 U_1^k(i,j)}{\partial y_1^2}-\grave{a}_{11y_1}(\vec{y}_{ij})\frac{\partial U_1^k(i,j)}{\partial y_1}-a_{12}(\vec{y}_{ij})\frac{\partial^2 U_1^k(i,j)}{\partial y_2^2}-\grave{a}_{12y_1}(\vec{y}_{ij})\frac{\partial U_1^k(i,j)}{\partial y_2}$$

$$+c_1(\vec{y}_{ij},t)U_1^{k+1}(i,j)-d(\vec{y}_{ij},t)U_2^{k+1}(i,j)=F_1^k(i,j)$$

$$U_{2t}^k(i,j)-a_{21}(\vec{y}_{ij})\frac{\partial^2 U_2^k(i,j)}{\partial y_1^2}-\grave{a}_{21y_1}(\vec{y}_{ij})\frac{\partial U_2^k(i,j)}{\partial y_1}-a_{22}(\vec{y}_{ij})\frac{\partial^2 U_2^k(i,j)}{\partial y_2^2}-\grave{a}_{22y_1}(\vec{y}_{ij})\frac{\partial U_2^k(i,j)}{\partial y_2}$$

$$+c_2(\vec{y}_{ij},t)U_2^{k+1}(i,j)-d(\vec{y}_{ij},t)U_1^{k+1}(i,j)=F_2^k(i,j)$$

Hence, they give

$$U_{1t}^k(i,j)-\sum_{r=1}^{2}\frac{\partial}{\partial y_r}\left[a_{1r}(\vec{y}_{ij})\frac{U_r}{y_r}\right]+c_1(\vec{y}_{ij})U_1^{k+1}(i,j)-d(\vec{y}_{ij})U_2^{k+1}(i,j)=F_1(\vec{y},t)$$

$$U_{2t}^k(i,j)-\sum_{r=1}^{2}\frac{\partial}{\partial y_r}\left[a_{2r}(\vec{y}_{ij})\frac{U_r}{y_r}\right]+c_2(\vec{y}_{ij})U_2^{k+1}(i,j)-d(\vec{y}_{ij})U_1^{k+1}(i,j)=F_2(\vec{y},t)$$

Thus the FFIDM is consistency

It is important to indicate here that the stability of the proposed couple FFIDM ((7)-(8)) is unconditional since the difference scheme which is used here is of the implicit form in time.

5. Numerical Examples

In this paper, the efficiency for the presented method is shown from the following numerical examples. It is important to refer here, since the CPSVC were studied in this work, were non-homogeneous, so it is possible to construct some examples which have an exact solution in order to compare the results and then to give an indicator about the efficiency for the presented method, and the method will be a basis to solve similar problem that they may have not exact solutions.

Example 1: Let $I=[0,1]$, $Q=\Omega\times I$, $\Omega(0,1)\times(0,1)$, $I=[0,1]$ then the CPSVC are

$$U_{1t}-\frac{\partial}{\partial y_1}\left[\left(y_1^2+1\right)\frac{\partial U_1}{\partial y_1}\right]-\frac{\partial}{\partial y_2}\left[\left(y_2^2+1\right)\frac{\partial U_1}{\partial y_2}\right]+\left(y_1^2y_2^2+1\right)U_1-\left(y_1^4+y_2^2+1\right)U_2=F_1(\vec{y},t)$$

$$U_{2t}-\frac{\partial}{\partial y_1}\left[\left(y_2^2+1\right)\frac{\partial U_2}{\partial y_1}\right]-\frac{\partial}{\partial y_2}\left[\left(y_1^2+1\right)\frac{\partial U_2}{\partial y_2}\right]+\left(y_1y_2^2+1\right)U_2+\left(y_1^4+y_2^2+1\right)U_1=F_2(\vec{y},t)$$

where the ICs and the BCs resp. are

$U_l(\vec{y},0)=U_l^0(\vec{y})=y_ly_l(1-y_l)(1-y_l)$, in Ω, for $l=1,2$

$U_l(\vec{y},t)=0$, on $\partial\Omega\times I$, for $l=1,2$

with $F_1(\vec{y},t)$ and $F_2(\vec{y},t)$ are given as

$$F_1(\vec{y},t)=y_1y_2(y_1-1)(y_2-1)\left[e^{-t}\left(y_1^2y_2^2-4\right)-\left(y_1^2+y_2^2+1\right)e^{-2t}\right]$$
$$-2y_1e^{-t}\left[y_1y_2(y_2-1)+(y_1-1)\left(y_2^2+1\right)+y_2^2(y_1-1)\right]-2y_2e^{-t}(y_2-1)\left(y_1^2+1\right)$$

$$F_2(\vec{y},t)=e^{-2t}\left[y_1^2\left(y_2^3(y_2-1)\right)-y_2^2\left(y_2^2+1\right)+\left(y_2-2\left(y_1^2+y_1+1\right)\right)-2y_2^2\left(y_2^2+y_2+1\right)-2(y_1+y_2)\right]$$
$$+e^{-t}\left[\left(y_2^2-y_2\right)\left(y_1^6-y_1^5\right)+y_1^2\left(y_2^4-y_2\right)-y_1\left(y_2^4-y_2^3+y_2^2+y_2\right)\right]$$

The Couple EXSO are

$U_1(\vec{y},t)=y_1y_2(1-y_1)(1-y_2)e^{-t}$ & $U_2(\vec{y},t)=y_1y_2(1-y_1)(1-y_2)e^{-2t}$

This problem is solved using the FFIDM for N = 9, NT = 20 and T = 1, the $NCS\vec{U}^n$, the EXSO $\vec{U}$ (at y_1, y_2) and the "absolute(ABS)" error between them are presented in Table (1) and were displayed in Figure (1) at the time $\hat{t}=0.5$. It is noted that the maximum of the ABS error ($MABSE$) $\left\|\vec{U}-\vec{U}^n\right\|_\infty=\max_{l=1,2}\left|U_l-U_l^n\right|$ is (0.0357).

Table 1: The EXSO and the NCS and the ABSE

y_1	y_2	EXSO	NS	ABSE	y_1	y_2	EXSO	NS	ABSE
0.1	0.1	0.0049	0.0013	0.0036	0.1	0.1	0.0030	0.0009	0.0021
0.3	0.1	0.0115	0.0013	0.0102	0.3	0.1	0.0070	0.0008	0.0061
0.5	0.1	0.0136	0.0014	0.0122	0.5	0.1	0.0083	0.0009	0.0074
0.7	0.1	0.0115	0.0013	0.0102	0.7	0.1	0.0070	0.0008	0.0062
0.9	0.1	0.0049	0.0007	0.0042	0.9	0.1	0.0030	0.0004	0.0026
0.1	0.3	0.0115	0.0011	0.0104	0.1	0.3	0.0070	0.0008	0.0061
0.3	0.3	0.0267	0.0016	0.0251	0.3	0.3	0.0162	0.0012	0.0150
0.5	0.3	0.0318	0.0020	0.0299	0.5	0.3	0.0193	0.0012	0.0181
0.7	0.3	0.0267	0.0021	0.0247	0.7	0.3	0.0162	0.0010	0.0152
0.9	0.3	0.0115	0.0014	0.0101	0.9	0.3	0.0070	0.0005	0.0064

0.1	0.5	0.0136	0.0012	0.0124	0.1	0.5	0.0083	0.0010	0.0073
0.3	0.5	0.0318	0.0018	0.0300	0.3	0.5	0.0193	0.0014	0.0179
0.5	0.5	0.0379	0.0022	0.0357	0.5	0.5	0.0230	0.0014	0.0216
0.7	0.5	0.0318	0.0023	0.0295	0.7	0.5	0.0193	0.0012	0.0181
0.9	0.5	0.0136	0.0016	0.0121	0.9	0.5	0.0083	0.0006	0.0076
0.1	0.7	0.0115	0.0011	0.0103	0.1	0.7	0.0070	0.0010	0.0060
0.3	0.7	0.0267	0.0018	0.0249	0.3	0.7	0.0162	0.0013	0.0149
0.5	0.7	0.0318	0.0021	0.0297	0.5	0.7	0.0193	0.0014	0.0180
0.7	0.7	0.0267	0.0021	0.0246	0.7	0.7	0.0162	0.0011	0.0151
0.9	0.7	0.0115	0.0014	0.0101	0.9	0.7	0.0070	0.0006	0.0063
0.1	0.9	0.0049	0.0009	0.0040	0.1	0.9	0.0030	0.0006	0.0024
0.3	0.9	0.0115	0.0017	0.0098	0.3	0.9	0.0070	0.0009	0.0060
0.5	0.9	0.0136	0.0019	0.0117	0.5	0.9	0.0083	0.0010	0.0073
0.7	0.9	0.0115	0.0017	0.0098	0.7	0.9	0.0070	0.0008	0.0061
0.9	0.9	0.0049	0.0008	0.0041	0.9	0.9	0.0030	0.0004	0.0026

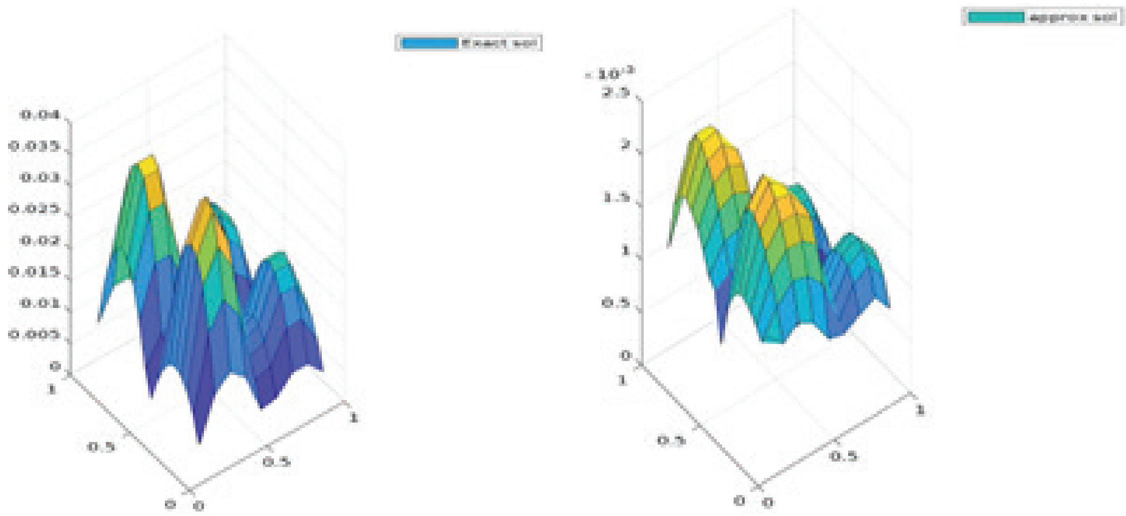

Figure 1: The EXSO and the NCS

Example 2: Let $I=[0,1], \Omega=(0,1)\times(0,1), Q=\Omega\times I$, then the CPSVC are

$$U_{1t}-\frac{\partial}{\partial y_1}\left[\left(y_1^2-2y_2+7\right)\frac{\partial U_1}{\partial y_1}\right]-\frac{\partial}{\partial y_2}\left[\left(y_1+1\right)\frac{\partial U_1}{\partial y_2}\right]+\left(y_1y_2+1\right)U_1-\left(2y_1^2+5y_2+11\right)U_2=F\left(\vec{y},t\right)$$

$$U_{2t}-\frac{\partial}{\partial y_1}\left[\left(y_2e^{y_1}\right)\frac{\partial U_2}{\partial y_1}\right]-\frac{\partial}{\partial y_2}\left[\left(y_2+1\right)\frac{\partial U_2}{\partial y_2}\right]+\left(y_1^2y_2^2\right)U_2+\left(2y_1^2+5y_2+11\right)U_1=F_2\left(\vec{y},t\right)$$

Where the ICs and the BCs resp. are

$$U_1\left(\vec{y},0\right)=U_1^0\left(\vec{y}\right)=2.7y_1\left(1-y_1\right)\left(1-y_2\right)\sin\left(\frac{y_2}{9}\right),\ in\ \Omega$$

$$U_2\left(\vec{y},0\right)=U_2^0\left(\vec{y}\right)=0,\ in\ \Omega$$

$$U_l\left(\vec{y},t\right)=0, \text{on } \partial\Omega\times I, \text{ for } l=1,2$$

with $F_1(\vec{y},t)$ and $F_2(\vec{y},t)$ are given as

$$F_1(\vec{y},t)=\sin\left(\frac{y_2}{9}\right)e^{\cos(t/9)}\ (y_2-1)\left[(y_1-1)\left(y_1(y_1y_2+1)-2y_1+y_1y_2\frac{e^{y_1}}{81}-\frac{y_1}{9}\sin(t/9)\right)-\left(4y_1^2-4y_2+14\right)\right]$$
$$-y_1(y_1-1)e^{\cos(t/9)}\left[e^{y_1}\left(2y_2\cos\left(\frac{y_2}{9}\right)+\sin\left(\frac{y_2}{9}\right)\right)-\frac{(y_2-1)}{9}\cos\left(\frac{y_2}{9}\right)\right]$$
$$-\left[\frac{\pi}{3}y_1(y-1)\left(2y_1^2+5y_2+11\right)\sin(y_1-1)\tan(t/9)\right]$$

$$F_2(\vec{y},t)=\frac{2}{3}\pi\tan(t/9)\left[\frac{1}{9}y_1(y_2+1)\sin\left(\frac{y_2}{9}\right)sin(1-y_1)-(y_1+1)(y_2-1)cos(y_1-1)\left(\cos\left(\frac{y_2}{9}\right)-1\right)\right]$$
$$+y_1(y_2-1)\tan(t/9)\left[\begin{array}{l} sin(y_1-1)\left(\frac{1}{243}(y_2+1)\cos\left(\frac{y_2}{9}\right)+\frac{1}{3}(y_1+1)\left(\cos\left(\frac{y_2}{9}\right)-1\right)\right)+ \\ \left(\frac{1}{27}\sin\left(\frac{y_2}{9}\right)sin(y_1-1)-cos(y_1-1)\left(\cos\left(\frac{y_2}{9}\right)-1\right)\right)\end{array}\right]$$
$$-sin(y_1-1)\left(\cos\left(\frac{y_2}{9}\right)-1\right)\left[\tan(t/9)(y_1+y_2+1)-\frac{1}{3}y_1(y_2-1)\left(\sqrt[9]{\tan^2(t/9)}+y_1^2y_2^2\tan(t/9)+\frac{1}{9}\right)\right]+$$
$$+y_1(y_1-1)(y_2-1)e^{\cos(t/9)}\sin\left(\frac{y_2}{9}\right)\left(2y_1^2+5y_2+11\right)$$

The Couple EXSO are

$$U_1(\vec{y},t)=y_1(1-y_1)(1-y_2)\sin\left(\frac{y_2}{9}\right)e^{\cos(-t/9)}$$

$$U_2(\vec{y},t)=\frac{1}{2}\pi y_1(1-y_2)\sin(1-y_1)\left(1-\cos\left(\frac{y_2}{9}\right)\right)\tan(-t/9)$$

This problem is solved using the FFIDM for $N=9$, $NT=20$ and $T=1$, the $NCS\vec{U}^n$, the EXSO $\vec{U}$ (at y_1, y_2) and the "absolute(ABS)" error between them are presented in Table (1) and were displayed in Figure (1) at the time $\hat{t}=0.5$. It is noted that the maximum of the ABS error $(MABSE)\left\|\vec{U}-\vec{U}^n\right\|_\infty=\max_{l=1,2}\left|U_l-U_l^n\right|$ is (0.0357).

Table 2: The EXSO and the NCS and the ABSE

		EXSO	NS 1.0e-03 *	ABSE			EXSO	NS1.0e-03 *	ABSE
0.1	0.1	0.0024	0.4929	0.0019	0.1	0.1	-0.0000	0.2195	0.0002
0.3	0.1	0.0057	0.2434	0.0055	0.3	0.1	-0.0000	0.2330	0.0002
0.5	0.1	0.0068	0.2499	0.0065	0.5	0.1	-0.0000	0.2462	0.0002
0.7	0.1	0.0057	0.2861	0.0054	0.7	0.1	-0.0000	0.1856	0.0002
0.9	0.1	0.0024	0.3340	0.0021	0.9	0.1	-0.0000	0.0597	0.0001
0.1	0.3	0.0057	0.5847	0.0051	0.1	0.3	-0.0000	0.2448	0.0002
0.3	0.3	0.0133	0.6257	0.0127	0.3	0.3	-0.0000	0.4972	0.0005

0.5	0.3	0.0158	0.6903	0.0151	0.5	0.3	-0.0000	0.5483	0.0006
0.7	0.3	0.0133	0.7777	0.0125	0.7	0.3	-0.0000	0.4193	0.0004
0.9	0.3	0.0057	0.7892	0.0049	0.9	0.3	-0.0000	0.1381	0.0001
0.1	0.5	0.0068	0.6545	0.0061	0.1	0.5	-0.0000	0.2773	0.0003
0.3	0.5	0.0158	0.7296	0.0151	0.3	0.5	-0.0000	0.5679	0.0006
0.5	0.5	0.0188	0.8234	0.0180	0.5	0.5	-0.0000	0.6308	0.0006
0.7	0.5	0.0158	0.9208	0.0149	0.7	0.5	-0.0000	0.4869	0.0005
0.9	0.5	0.0068	0.8778	0.0059	0.9	0.5	-0.0000	0.1633	0.0002
0.1	0.7	0.0057	0.5175	0.0052	0.1	0.7	-0.0000	0.2235	0.0002
0.3	0.7	0.0133	0.6129	0.0127	0.3	0.7	-0.0000	0.4610	0.0005
0.5	0.7	0.0158	0.7127	0.0151	0.5	0.7	-0.0000	0.5152	0.0005
0.7	0.7	0.0133	0.7879	0.0125	0.7	0.7	-0.0000	0.4006	0.0004
0.9	0.7	0.0057	0.6933	0.0050	0.9	0.7	-0.0000	0.1365	0.0001
0.1	0.9	0.0024	0.2352	0.0022	0.1	0.9	-0.0000	0.0922	0.0001
0.3	0.9	0.0057	0.3419	0.0053	0.3	0.9	-0.0000	0.1915	0.0002
0.5	0.9	0.0068	0.4308	0.0063	0.5	0.9	-0.0000	0.2150	0.0002
0.7	0.9	0.0057	0.4620	0.0052	0.7	0.9	-0.0000	0.1682	0.0002
0.9	0.9	0.0024	0.3340	0.0021	0.9	0.9	-0.0000	0.0581	0.0001

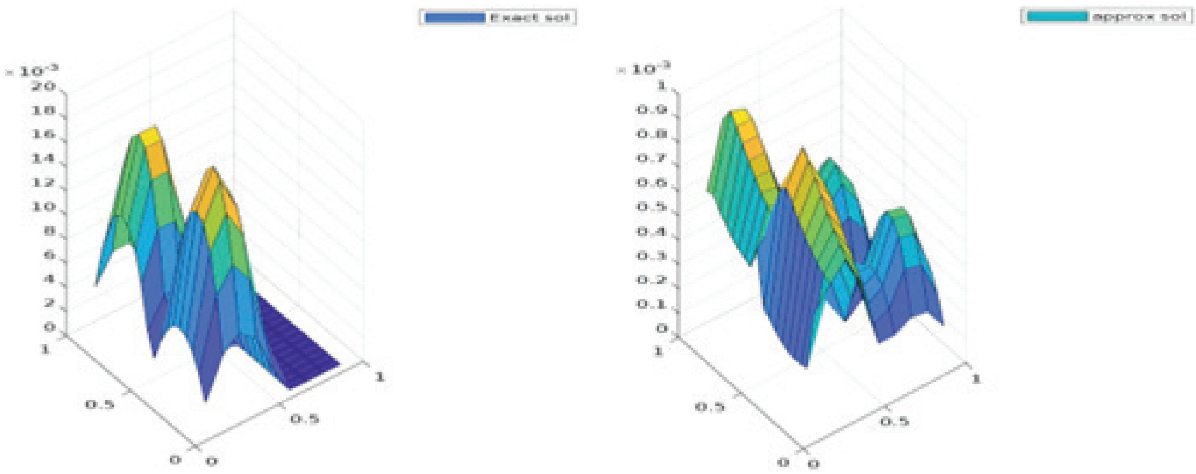

Figure 2: The EXSO and the NSO

Conclusions

The consistency (the stability) of this method is proved (is ensured). The FFIDM is applied successfully to solve the CPSVC. The results were obtained through solving the two illustrative examples were found at each discrete value (y_1 and y_2) for the space variable (N = 9) and at each discrete value of time t_j in [0, 1] (with NT = 20), but although the tables contain the couple solutions only for $t = 0.5$ and for the half of the space variable of space values only this back to save the space.

From the results in Tables 1,2 and in the Figures 1,2, we saw that the absolute maximum error between the EXSO and the NCS for the considered problems show the efficiency of the method, although the space variable is discretized only for ten grid (N = 9) and the variable of time for NT = 20. The CLS was solved by utilizing the GEM, and we conclude that this method is accurate and suitable for solving such systems.

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