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Von-Neumann regular Q-Algebras and Q-Digraphs

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Abstract

This study presents the notions of Von-Neumann regular Q-algebra and Q-digraph. Given a Q-algebra X, the corresponding graph, indicated by $\Gamma(\mathbb{X})$, is a directed graph with vertices that correspond to elements of X. For two different elements a, $b \in X$, an Arc from a to b (written as $a \to b$) exists if and only if $a \,\Delta\, b = 0$, where $a \,\Delta\, b = (b * a) * a$. We elaborate these ideas and offer examples. The paper also, analyze the Q-algebra $(\mathbb{Z}_n; -, 0)$ and indicates that it is a left Von-Neumann regular Q-algebra. In addition, features of the Q-digraph corresponding to the Q-algebra $(\mathbb{Z}_n; -, 0)$ will be examined. The main conclusion of this research is that the digraph linked. The principal finding of this paper will shed light on the digraph associated with $\mathbb{Z}_{(4.2^{n-1})}$ forms a tree digraph for all $n \geq 1$.

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Key words and Phrases: Q-algebra, Von-Neumann regular element, diagraph, tree diagraph, connectedness.

1. Introduction

BCK-algebra and BCI-algebra are two abstract algebraic structures that were presented by Imai and Is´eki [7,8]. The BCK-algebra class is closely related to the broader BCI-algebra class, as is well known. The BCH-algebra, which includes the BCI-algebra, was suggested by Hu and Li [9]. D-algebra was created by Neggers and Kim [10] and is defined as follows:

i. $\alpha * \alpha = e$,
ii. $e * \alpha = e$,
iii. If $\alpha * \beta = e$ and $\beta * \alpha = e$, then $\alpha = \beta$.

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This idea serves as a substantial extension of BCK-algebra. Researchers have explored multiple links between d-algebra and BCK-algebra, as well as examined how these structures relate to oriented digraphs. Meanwhile, Jun, Roh, and Kim [6] presented the concept of BH-algebra as follows:

i. $\alpha * \alpha = e$,
ii. $\alpha * e = \alpha$;
iii. If $\alpha * \beta = e$ and $\beta * \alpha = e$, then $\alpha = \beta$.

The BH-algebra structure broadens the scope of BCH, BCI, and BCK-algebras. It has been established that a maximal ideal exists within bounded BH-algebras. The concept of Q-algebra, introduced by Neggers, Ahn, and Kim, offers a further generalization encompassing BCH, BCI, and BCK-algebras, and also generalizes certain theorems known from BCI-algebra theory. In addition, their introduction of "quadratic" Q-algebras led to the proof that every quadratic Q-algebra $(A\,;\,e)$, where $e \in X$, can be expressed as $\alpha * \beta = \alpha - \beta + e$, where $\alpha, \beta \in A$ and A represents a field with $|A| \geq 3$, thus revealing a specific linear characteristic in the product.

In [11], Taloukolaei and Sahebi presented the idea of the Von Neumann regular graph made from a ring. They designated the related Von Neumann regular graph as $G_{Vnr^+}(R)$ and defined Vnr(R) as the collection of all regular elements inside R. Two vertices are linked exactly when their total is a regular element in R. The vertices of this graph correspond to elements of R. Their study focused on fundamental characteristics of this graph, such as its diameter, connectedness, planarity, and girth. Using Vizing's theorem, they classified the Von Neumann regular graph as a first-class graph. In related work, Jun and Lee [12] defined graphs associated with CK-algebras and BCI-algebras. For a CK-algebra or BCI-algebra A, the graph $\Gamma(A)$ has vertices representing elements of A, with adjacency between vertices a and b determined by the condition $l(a, b) = 0$, where $l(A) = \{h \in A : h * h' = 0 \text{ for all } h' \in A\}$. They provided several characterizations of these graphs, including the fact that $\Gamma(A)$ is connected when A is a BCI-algebra, and that every other nonzero vertex is adjacent to the vertex 0. Furthermore, graphs related to *KU*-algebras and *UP*-algebras were introduced in [5,6]. For A being either a *UP*-algebra or *KU*-algebra, the graph $G(A)$ is defined with vertices as elements of A, where two vertices a and b are adjacent if $a \,\Delta\, b = 0$, where $a \,\Delta\, b = (b * a) * b$. These studies presented numerous properties and characterizations of such graphs. Notably, in both cases, the diameter satisfies $\text{diam}(G(A)) \leq 3$, and the graph is connected. Moreover, if A and V are two such algebras and $G(A) \cong G(V)$, then their equivalence class graphs also satisfy $G_E(A) \cong G_E(V)$, where $G_E(A)$ denotes the graph formed by the equivalence classes of A. The current work extends these ideas to Q-algebras, a generalization that incorporates BCH, BCI, and BCK-algebras among others. The directed graphs (Q-digraphs) associated with Q-algebras have vertices as elements with arcs defined by a zero operation condition reflecting the Q-algebra structure. The manuscript shows that these digraphs capture finer structural features such as tree and anti-arborescence forms for specific classes $\mathbb{Z}_{(4.2^{n-1})}$. This provides a broad unifying framework for understanding algebraic regularity and relational graph structures beyond previous specific algebra classes.

Building on existing research, this study extends the notion of Von Neumann regular elements from ring theory to the framework of Q-algebras. Additionally, we introduce the new concept of Q-graphs. Let $(X; *, 0)$ be a Q-algebra. An element $a \in X$ is called a left Von Neumann regular element if there exists some $b \in X$ satisfying the condition $a = (a * b) * a$. Similarly, a is termed a right Von Neumann regular element if there exists $b \in X$ such that $a = a * (b * a)$. In both cases, the element b is referred to as a Von Neumann inverse of a. An element a in a Q-algebra $(X, *, 0)$ is *Von Neumann regular* if there exists $b \in X$ simultaneously satisfying the equations $a = (a * b) * a$ and $a = a * (b * a)$. A Q-algebra is called left (respectively right) Von Neumann regular when all its elements satisfy the first (respectively second) equation, and is fully Von Neumann regular when both conditions hold throughout. We associate to any Q-algebra X a directed graph $\Gamma(X)$ whose vertex set is X itself, with directed edges $x \to y$ between distinct elements existing precisely when $(y * x) * x = 0$. This work investigates the characterization of Von Neumann regular elements in the Q-algebras $\mathbb{Z}_n$ for selected integers n, along with the structural properties of their corresponding digraphs $\Gamma(\mathbb{Z}_n)$. In Section 3, Our study examines three key aspects of Q-algebra structures. First, we analyze the modular arithmetic properties of $(\mathbb{Z}_n;-,0)$, where the congruence relation $((n-k)-(n-i)-(n-j)) \equiv 0 \pmod{n}$ holds for sequential indices $j = 1,2,3,\ldots$ until termination when $n-k \in \{0,1\}$, with $i = j + 1$ and $k = i + j$. Next, we characterize the regularity properties, proving that $(\mathbb{Z}_n;-,0)$ constitutes a left-regular Q-algebra under the Von Neumann criterion. Finally, we explore associated digraph structures, showing that for $\Gamma(\mathbb{Z}_n)$ representing $\mathbb{Z}_n$, the specific case $\mathbb{Z}_{4\cdot 2^{n-1}}$ always produces an arborescence for $n \geq 1$. In disconnected cases, components containing vertices adjacent to 0 form subalgebras, with exactly $n/2$ such components existing. These exhibit distinct patterns based on parity: forming arithmetic progressions when odd, or requiring successive halving until oddness when even.

2. Preliminaries

This section provides fundamental definitions and associated concepts.

Definition 2.1: A nonempty set X, a distinct element 0, and a binary operation "*" on X that meets the following requirements constitute a Q-algebra:

- For every element x in X, applying "*" to x with itself yields 0,
- For any x in X, the operation of x with 0 returns x,
- The operation satisfies a modified associativity: for all x, y, z in X, the equality $(x * y) * z = (x * z) * y$ holds.

Declaring $x \leq y$ precisely when $x * y = 0$ allows us to construct a partial order $\leq$ on X.

Example 2.2. Consider the set of all integers $\mathbb{Z}$ equipped with binary subtraction and zero element. The triple $(\mathbb{Z},-,0)$, where "–" denotes the standard integer subtraction operation and 0 represents the additive identity, naturally forms a Q-algebra structure. This construction emerges from interpreting the subtraction operation as the fundamental binary relation and zero as the nullary operation satisfying the Q-algebra axioms.

Example 2.3. Examine the next table, which defines the operation "*" on the finite set $X = \{0, 1, 2, 3\}$:

Table 1: Operation table for Example 2.3

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

One can easily confirm that the structure $(X; *, 0)$ adheres to the defining axioms of a Q-algebra.

Definition 2.4. For a given Q-algebra $(X; *, 0)$, a non-empty subset $\mathcal{I} \subseteq X$ constitutes an ideal if it satisfies:

(1) The zero element belongs to $\mathcal{I}$, i.e., $\mathbf{0} \in \mathcal{I}$,
(2) For all elements $a, b \in X$, the membership $a * b \in \mathcal{I}$ combined with $b \in \mathcal{I}$ implies $a \in \mathcal{I}$.

Example 2.5. Within the Q-algebra $(\mathbb{Z}_{12}; -, 0)$, the set $\mathcal{I} = \{0, 4, 8\}$ forms an ideal.

Definition 2.6: An element x of a ring $\mathcal{R}$ exhibits Von Neumann regularity if there exists $y \in \mathcal{R}$ such that $x = xyx$. When this property holds for every element of $\mathcal{R}$, we say $\mathcal{R}$ is a Von Neumann regular ring.

Corollary 2.7. *For a regular element a in* $\mathbb{Z}_n$, *the element* $a^{\varphi(n)-1}$ *acts as a Von Neumann inverse of a in* $\mathbb{Z}_n$. *In particular,*

$$a^{-1} \equiv a^{\varphi(n)-1} \pmod{n}.$$

A directed graph, often called a digraph, G is composed of three fundamental parts: a set of vertices $V = r_1, r_2, \ldots$, a collection of edges $E = t_1, t_2, \ldots$, and a function ψ which maps each edge to a specific ordered pair of vertices (r_i, r_j). In graphical terms, vertices appear as points, and edges are shown as arrows pointing from vertex r_i toward vertex r_j. For example, Figure 1 illustrates a digraph that includes five vertices connected by ten edges. Another term used for digraphs is oriented graphs.

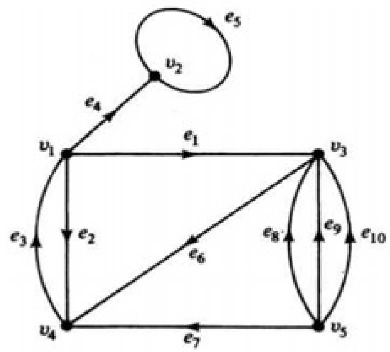


Figure 1: The digraph containing five vertices and ten edges.

In a directed graph, or digraph, each edge has a direction, meaning it originates from one vertex and points toward another. The vertex r_i from which an edge t_k starts is called the initial vertex of t_k, while the vertex t_j where t_k ends is known as the terminal vertex.

The count of edges that emanate from a vertex r_i is referred to as its out-degree (also called out-valence or outward demidegree), denoted by $d^+(r_i)$. Conversely, the number of edges that arrive at r_i is called its in-degree (also known as invalence or inward demidegree), denoted by $d^-(r_i)$. For example, in Figure 1:

$$d^+(r_1) = 3 \text{ and } d^-(r_1) = 1,$$
$$d^+(r_2) = 1 \text{ and } d^-(r_2) = 2,$$
$$d^+(r_5) = 4 \text{ and } d^-(r_5) = 0.$$

It is a fundamental property of digraphs that the total sum of all vertices' out-degrees equals the total sum of all in-degrees. Both sums correspond to the total number of edges in the digraph, expressed mathematically as:

$$\sum_{i=1}^{n} d^+(r_i) = \sum_{i=1}^{n} d^-(r_i).$$

A vertex is called isolated if it has no incoming or outgoing edges, meaning both its in-degree and out-degree are zero. A vertex r is termed pendant if the sum of its in-degree and out-degree equals one, that is,

$$d^+(r) + d^-(r) = 1.$$

Definition 2.8. [13] A tree digraph is a directed graph that is connected and free from any form of cycles, meaning it does not contain directed cycles or semicircuits. For a tree digraph with n vertices, the number of edges is always exactly $n - 1$.

3. Some properties of $\mathbb{Z}_n$ as a *Q*-algebra

This section delves into the characteristics of a specific type of Q-algebra, exploring examples, theorems, and definitions to understand its properties.

Example 3.1. Suppose that $\mathbb{Z}_n$ is the set of integers modulo *n*. Then $(\mathbb{Z}_n;-,0)$ form a Q-algebra.

The following theorem establishes essential modular arithmetic relations in the Q-algebra $\mathbb{Z}_n$, which form the foundational structure for later results on Von-Neumann regularity

Theorem 3.2. *Let* $(\mathbb{Z}_n;-,0)$ *be* **Q** – *algebra with subtraction taken modulo n. For integers p, r, t satisfying* $r = t + 1$ *and* $p = r + t$, *we have* $((n - p) - (n - r) - (n - t)) \bmod n = 0$. *Stop the process when* $(n - p) \in \{0, 1\}$.

Proof. By mathematical indication. First step: suppose that $t = 1$. Then

$$((n-3)-(n-2)-(n-1)) \bmod n$$
$$= (n-3-n+2-n+1) \bmod n = -n \bmod n = 0$$

So, it is true for $t = 1$. Second step: Asume that it is true for $t = y$. Then Then $((n - p) - (n - r) - (n - y)) \bmod n = 0$. Where $r = y + 1$ and $p = r + t$. $y = 1, 2, 3 \ldots (n - p) = 0$ or 1 . We need to prove for $w + 1$. Then $((n - p) - (n - r) - (n - (y + 1))) \bmod n = (n - 2y - 3 - n + y + 2 - n + y + 1) \bmod n = -n \bmod n = 0$. □

Example 3.3. Let $(\mathbb{Z}_{10}; -, 0)$ be a Q– algebra. Then by Theorem 3.2:

$$(7-8-9) \bmod 10 = -10 \bmod 10 = 0$$
$$(5-7-8) \bmod 10 = -10 \bmod 10 = 0$$
$$(3-6-7) \bmod 10 = -10 \bmod 10 = 0$$
$$(1-5-6) \bmod 10 = -10 \bmod 10 = 0$$

Example 3.4. Let $(\mathbb{Z}_{11}; -, 0)$ be a Q - algebra. Then by Theorem 3.2 we have

$$(8-9-10) \bmod 11 = -11 \bmod 11 = 0$$
$$(6-8-9) \bmod 11 = -11 \bmod 11 = 0$$
$$(4-7-8) \bmod 11 = -11 \bmod 11 = 0$$
$$(2-6-7) \bmod 11 = -11 \bmod 11 = 0$$
$$(0-5-6) \bmod 11 = -11 \bmod 11 = 0$$

Corollary 3.5. *Consider the Q-algebra* $(\mathbb{Z}_n;-,0)$. *Then the number of* $((n - p) - (n - r) - (n - t)) \bmod n = 0$ *in* $\mathbb{Z}_n$ *is* $\frac{n-1}{2}$, *if n is odd and is* $\frac{n-2}{2}$, *if n is even.*

Corollary 3.6. *Consider the* $\mathrm{Q}-\text{algebra}(\mathbb{Z}_n;-,0)$. *Then* $((n-t)+(n-r)+(n-p)) \bmod n = 0$ *where* $t = 1, 2, 3, \ldots (n - p) = 0$ *or* 1, $r = t + 1$ *and* $p = r + t$.

Definition 3.7. Suppose that $(X; *, 0)$ is a $\mathbb{Q}$ – algebra and $\varnothing \neq Y \subseteq X$, then $(Y; *, 0)$ is a sub $\mathbb{Q}$ – algebra, if $(Y; *, 0)$ is $\mathbb{Q}$ – algebra.

Example 3.8. Let $(\mathbb{Z}_{12}; -, 0)$ be the $\mathbb{Q}$ – algebra with subtraction taken modulo 12. the set $Y = \{0,2,4,6,8,10\}$ is a sub $\mathbb{Q}$ – algebra of $(\mathbb{Z}_{12}; -, 0)$.

In the following result we give a characteraization of sub $\mathbb{Q}$ – algebra

Theorem 3.9. *Let $(X ; *, 0)$ be Q - algebra and let $\varnothing \neq S \subseteq X$. Then $(S ; *, 0)$ is a sub Q - algebra of X if and only if the following conditions are satisfied:*

a) $0 \in S$,
b) $(x * y) * z \in S$ for all $x, y, z \in S$.

Proof. Suppose that S is a sub Q - algebra, then imples $0 \in S$, then $(x * y) * z = (x * y) * z$, so $(x * y) * z \in S$.

Coversely, assume that $S \subseteq X$, $0 \in S$ and $(x * y) * z \in S$. Let $x, y, z \in S$, then we get $x * x = 0$ and $x * 0 = x$ for all x. Since $(x * y) * z = (x * y) * y \in S$. So $(S ; *, 0)$ is a sub Q - algebra of X. □

Theorem 3.10. *Let $(X_1; -, 0)$ and $(X_1; -, 0)$ be two sub Q– algebras of a Q-algebra $(\mathbb{Z}_n;-,0)$. Then the intersection of X_1 and X_2 also is sub Q - algebra.*

Proof. Suppose that X_1 and X_2 are sub Q- algebra of $\mathbb{Z}_n$. So $0 \in X_1 \cap X_2 \neq \varnothing$. Let $x,y,z \in X_1 \cap X_2$. So $x, y, z \in X_1$ and $x, y, z \in X_2$. Since X_1 and X_2 are sub Q– algebra, so $(x - y) - z \in X_1 \cap X_2$. Therefore $(X_1 \cap X_2;-,0)$ is a sub Q– algebra. □

Example 3.11. Let $X_1 = \{0, 4, 8\}$ and $X_1 = \{0, 2, 4, 6, 8, 10\}$ be two sub Q– algebra of $(\mathbb{Z}_{12}; -, 0)$. Then $X_1 \cap X_2$ is sub Q– algebra of a Q– algebra $(\mathbb{Z}_{12}; -, 0)$.

Theorem 3.12. Union of two Sub Q-algebra is not necessarily a sub Q-algebra. Let $(\mathbb{Z}_n;-,0)$ be a Q-algebra, and let X_1 and X_2 be two sub Q-algebras of $\mathbb{Z}_n$. The union $X_1 \cup X_2$ of these two sub Q- algebras is a sub Q- algebra of $\mathbb{Z}_n$ if $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$.

Proof. Suppose $X_1 \subseteq X_2$. Then $X_1 \cup X_2 = X_2$, which is a sub Q– algebra of $\mathbb{Z}_n$. Similary, if $X_2 \subseteq X_1$. Then $X_1 \cup X_2 = X_1$, which is a sub Q- algebra of $\mathbb{Z}_n$. □

Example 3.13. Let $X_1 = \{0,4,8\}$ and $X_2 = \{0,3,6,9\}$ be two sub Q– algebra of $(\mathbb{Z}_{12}; -, 0)$. Then $X_1 \cup X_2$ is not sub Q– algebra of a Q– algebra $(\mathbb{Z}_{12}; -, 0)$. Since $X_1 \cup X_2 = \{0,3,4,6,8,9\}$, but $(0-3)-4 = -7 \bmod 12 = 5 \notin X_1 \cup X_2$. Since $X_1 \subseteq X_2$. Then $X_1 \cup X_2 = X_2$ is a sub Q– algebra of a Q– algebra Z_{12}.

Definition 3.14. Consider two Q-algebras, $(X_1;*_1,0)$ and $(X_2;*_2,0)$. The product of X_1 and X_2, denoted as $X_1 \times X_2$, is defined as the set of all ordered pairs (x_1, x_2) where x_1 is an element of x_1 and x_2 is an element of X_2. This set is equipped with the operation defined by: $(x_1,x_2) * (y_1,y_2) = (x_1 *_1 y1, x_2 *_2 y2)$.

Example 3.15. Let $(\mathbb{Z}_3;-,0)$ and $(Z_4;-,0)$ are Q– algebra. Then $(\mathbb{Z}_3 \times \mathbb{Z}_4;-,0)$ is Q– algebra.

4. Von Neumonn regular Q– algebra

In this section ,we define Von- Neumonn regular element for Q– algebra and left (right) Von-Neumonn regular Q– algebra.

Definition 4.1. Let $(X ; *, 0)$ be a Q-algebra and let $a \in X$. The element a is called left Von Neumonn regular element if there exist an element w in X such that $a = (a * w) * a$. Similarly, a is called right Von-Neumonn regular element if there exist an element w in X such that $a = a * (w * a)$. Any such w is called a Von Neumonn inverse of a. If there exists w in X such that $a = (a * w) * a = a * (w * a)$, then a is a Von-Neumonn regular element in X. The Q– algebra X is said to be left (respectively right) Von-Neumonn regular if every element of X is left (respectively right) Von Neumonn regular. If X is both left and right Von Neumonn regular, thenit is called Von Neumonn regular Q– algebra.

Example 4.2. Consider the finite set $X = \{0,1,2,3\}$ and "*" is an operation defined in X as the following table:

Table 2: The operation for Example 4.2.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then it is easy to show that (X; *, 0) form a Q– algebra. The left Von-Neumonn regular element in (X; *, 0) is only zero element, but right Von-Neumonn regular element is all the set X.

Example 4.3. Consider the finite set $X = \{0,1,2\}$ and "*" is an operation defined in X as the following table:

Table 3: The operation for Example 4.3.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then again (X; *, 0) form a Q– algebra. The elements of the set X are left and right Von-Neumonn regular elements, but X is not Von-Neumonn regular Q– algebra, since $(a * w) * a \neq a *(w * a)$ for all a, w in X.

Remark 4.4. Consider the Q– algebra $(\mathbb{Z}_n;-,0)$. Then its left Von-Neumonn regular elements are $\{0,1,\ldots,n-1\}$.

Example 4.5. To show $(\mathbb{Z}_5\,; -, 0)$ is a left Von-Neumonn regular Q– algebra, let

$$a = 0 = ((0-0)-0) \bmod 5 = 0$$
$$a = 1 = ((1-4)-1) \bmod 5 = -4 \bmod 5 = 1$$
$$a = 2 = ((2-3)-2) \bmod 5 = -3 \bmod 5 = 2$$
$$a = 3 = ((3-2)-3) \bmod 5 = -2 \bmod 5 = 3$$
$$a = 4 = ((4-1)-4) \bmod 5 = -1 \bmod 5 = 4.$$

Corollary 4.6. *The Q– algebra $(\mathbb{Z}_n;-,0)$ is a left Von-Neumonn regular Q– algebra.*

The following result identifies explicit Von-Neumann inverses for left regular elements, a key step to understanding the fine structure of regular elements in $\mathbb{Z}_n$.

Theorem 4.7. *If a is a left regular element in $\mathbb{Z}_n$, then $(n - a)$ is a Von-Neumann inverse for a.*

Proof. Let $a \in \mathbb{Z}_n$ be a Von Neumann regular element. Then

$$((a-(n-a))-a) \bmod n = -(n-a) \bmod n$$
$$= n \bmod n + a \bmod n = a \bmod n = a.$$

□

Proposition 4.8. *Consider that $a \in \mathbb{Z}_n$, and w is a Von-Neumann inverse for a.*

Then

$$a - w = 0$$

Proof. If w form a Von-Neumann inverse for a. Since by Theorem 4.7. $w = n - a$. Then $a - w = a - n - a = a - a - n = -n \bmod n = 0$. □

Proposition 4.9. *Consider the Q– algebra X. Then we have the following equivalent statements for every a in X.*

(1) If a is regular element,
(2) $a = (a * a) * b$, where b is Von-Neumann inverse for a.

Proof. (1) ⇒ (2) Let a be a regular element. Then there exist $b \in X$ such that $a = (a * b) * a = a * (b * a)$, then $a = (a * b) * a = (a * a) * b$.

(2) ⇒ (1) Suppose that b is Von Neumann inverse for a. Then $a = (a * a) * b$, we have $a = (a * a) * b = (a * b) * a$. Thus a is Von-Neumann regula element. □

Corollary 4.10. *Consider an element a in $(\mathbb{Z}_n; -, 0)$ that is Von Neumann regular. Then, there is one and only one element b in $\mathbb{Z}_n$ satisfying the conditions $(a - b) - a = a$ and $(b - a) - b = b$.*

5. Digraph of Q– algebra

Definition 5.1. Given a Q– algebra X, its graph $\Gamma(\mathbb{X})$ is defined as a directed graph whose vertices correspond to the elements of X. For any two distinct elements $x, y \in X$, there is an arc from x to y (denoted $x \to y$) if and only if the expression $x \Delta y$ equals zero, where $x \Delta y$ is given by $(y * x) * x$. This type of directed graph is known as a Q– digraph.

Example 5.2. Consider the set $X = \{0, 1, 2\}$ accompanied by the table below:

Table 4: The operation for Example 5.2.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X; *, 0)$ is a Q– algebra. The digraph of $\Gamma(\mathbb{X})$ is:

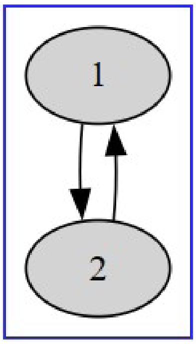


Figure 2: The diagraph for Example 5.2.

Table 5: The operation for Example 5.3.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Example 5.3. Define the set $X = \{0, 1, 2, 3\}$ with the operation specified in Table 5. Equipped with this operation and the element 0, the structure $(X; *, 0)$ forms a Q-algebra. The directed graph $\Gamma(\mathbb{X})$ corresponding to this algebraic system is illustrated in Figure 3.

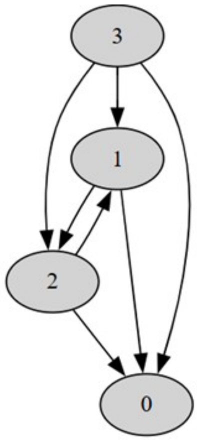


Figure 3: The diagraph of Example 5.3.

Remark 5.4. Let $\Gamma(\mathbb{X})$ be a digraph of $\mathbb{Z}_n$, then the following are true:

- If n is odd, then every vertex has only one indegree and one outdegree.
- If n is even, then every odd vertex has only one outdegree and zero indegree, but every even vertex has two indegree and one outdegree.

The main structural property of the Q-digraph associated with $\mathbb{Z}_{4.2^{n-1}}$ reveals a tree-like organization important for understanding connectivity and acyclicity in these graphs. In the following results, we investigate these properties.

Theorem 5.5. *Let $\Gamma(\mathbb{Z}_n)$ be a digraphs of $\mathbb{Z}_n$. Then $\Gamma\left(\mathbb{Z}_{(4.2^{n-1})}\right)$ is a tree digraph, for all $n \geq 1$.*

Proof. We use mathamtical induction. For $n = 1$, then $\Gamma(\mathbb{Z}_{4.2^{n-1}}) = \Gamma(\mathbb{Z}_4)$, is a tree digraph. Suppose that it is true for $n = k$, means that $\Gamma(\mathbb{Z}_{4.2^{k-1}})$ is tree digraph. Now for $n = k + 1$, we get $\left(\mathbb{Z}_{4.2^{n}}\right) = \left(\mathbb{Z}_{4\cdot 2^{k+1-1}}\right) = \mathbb{Z}_{4.2^k}$. In $\Gamma(\mathbb{Z}_{4.2^k})$ we have one vertex in bottom from the digraph of degree one and the other vertices in the middle of digree three until in the top of the digraph all 2.2^{k-1} vertices of degree one which are $V = \{v_1, v_2, \ldots, v_{2.2^{k-1}}\}$ and $4.2^{k-1} = 2.\ (4.2^{k-1})$ and by adding 4.2^{k-1} vertices which are $\{u_1, u_2, \ldots, u_{4.2^{k-1}}\}$ then $U = \{u_1, u_2, \ldots, u_{2.2^{k-1}}, u_{2.2^{k-1}+1}, \ldots, u_{4.2^{k-1}}\}$. We see that in the digraph of $\Gamma(\mathbb{Z}_{4.2^{k-1}})$, the number of vertices are $4.2^{k-1} + 4.2^{k-1} = 4.2^k$ and we have for $\Gamma(\mathbb{Z}_{4.2^k})$ each two vertices of U are adjacent with only one vertices of V. Thus by this sequence u_1, u_2 adjacent with v_1 also u_3, u_4 are adjacent with v_1 and finally $u_{4.2^{k-1}-1}, u_{4.2^{k-1}}$ are adjancent with $v_{2.2^{k-1}}$, then we get that $\Gamma(\mathbb{Z}_{4.2^k})$ is tree digraph. Suppose that we have a cyclic digraphs $C = w_1, w_2, w_3, \ldots w_m, w_1$, which implies that degree $(w_1) = 2$ in cyclic. Hence we have three cases:

Case one: If w_1 in the bottom of digraph then degree of $w_1 = 3$ it is a contradiction.

Case two: If $w_1 \in \{u_1, u_2, \ldots u_{2.2^{k-1}}\}$, then the degree of w_1 is $4 \leq \deg(w_1) \leq 5$ which is impossible.

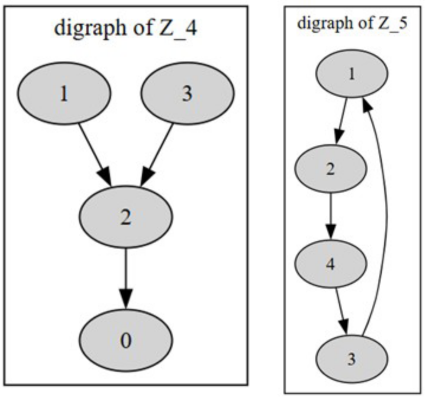
Case three: Suppose that $w \in V = \{v_1, v_2, \ldots v_{2.2^{k-1}}\}$, means that it is in the top of the digraph and $2 \leq \deg(w) \leq 3$ which is a contradiction. So the digraphs is not cycle. □

Lemma 5.6. The distance in $\Gamma\left(\mathbb{Z}_{(4.2^{n-1})}\right)$ from leaf to root is $d\left(\Gamma\left(\mathbb{Z}_{(4.2^{n-1})}\right)\right) = 2 + (n-1)$

Proof. We use mathamtical induction. We see that, for $n = 1$ is true. Then $d\left(\Gamma\left(\mathbb{Z}_4\right)\right) = 2 = 2 - (1-1)$. Assume that for $n = k$ is also true $d\left(\Gamma\left(\mathbb{Z}_{4.2^{k-1}}\right)\right) = 2 + (k-1)$. For $n = k + 1$, we must show that $d\left(\Gamma\left(\mathbb{Z}_{4.2^k}\right)\right) = 2 + (k)$. In $\left(\Gamma\left(\mathbb{Z}_{4.2^k}\right)\right)$ we have one vertex in bottom from the digraph of degree one and the other verteces are in the middle of digree three. However, in the top of the digraph all 2.2^{k-1}

vertices of degree one. Now, $V=\{v_1,v_2,\ldots,v_{2.2^{k-1}}\}$ and $4.2^k=2.(4.2^{k-1})$ and by adding 4.2^{k-1} vertices which are $\{u_1,u_2,\ldots,u_{4.2^{k-1}}\}$ then $U=\{u_1,u_2,\ldots,u_{2.2^{k-1}},u_{2.2^{k-1}+1},\ldots,u_{4.2^{k-1}}\}$. On the other hand, the digraph of $\Gamma(\mathbb{Z}_{4.2^{k-1}})$, then the number of vertices are $4.2^{k-1}+4.2^{k-1}=4.2^k$. In $\mathbb{Z}_{4.2^k}$ we have each two vertices of U are adjacent with only one vertice of V by this sequence u_1, u_2 adjacent with v_1 also u_3, u_4 are adjacent with v_2. Finally $u_{4.2^{k-1}-1},u_{4.2^{k-1}}$ are adjancent with $v_{2.2^{k-1}}$, so we get $d\left(\Gamma\left(\mathbb{Z}_{4.2^k}\right)\right)=2+(k-1)+1=2+k$.

Example 5.7. Let $(\mathbb{Z}_n;-,0)$ be the Q- algebra with subtraction taken modulo n. the diagraph of this Q- algebra is shown in Figures 4 and 5 for n = 4,5,20.



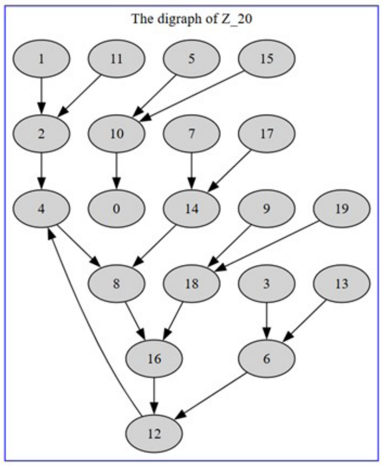


Figure 4: The diagraph of $\Gamma(\mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_5)$. Figure 5: The diagraph of $\Gamma(\mathbb{Z}_{20})$

Theorem 5.8. *Let $G=\Gamma(\mathbb{Z}_n)$ a digraphs of $(\mathbb{Z}_n;-,0)$. If G is disconnected, then every components that contain a vertex has Arc with 0 vertex, is sub Q- algebra. The set of components that has Arc with 0 is $\frac{n}{2}=a$, if a is odd then $G_1=\{0,a,2a,3a,\ldots ra=n\}$, where $r\in N$. If a is even then we keep dividing the result by 2 until we get odd number.*

Proof. Suppose that G is disconnected digraph of $\Gamma(\mathbb{Z}_n)$ and assume G_1 is the components that has Arc with 0 vertex. So $0\in G_1\neq\varnothing$, we most prove that $(x-y)-z\in G_1$ for all $x,y,z\in G_1$, since $x=r_1a, y=r_2a$, and $z=r_3a$, where $r_1,r_2,r_3\in N$, then $(x-y)-z=(r_1a-r_2a)-r_3a=(r_1-r_2)a-r_3a=((r_1-r_2)-r_3)a\in G_1$. Thus G_1 is a sub Q- algebra. □

Example 5.9. Let $(\mathbb{Z}_4;-,0),(\mathbb{Z}_8;-,0),(\mathbb{Z}_{16};-,0)$ are a Q– algebras. The digraphs of them are presented in Figure 6.

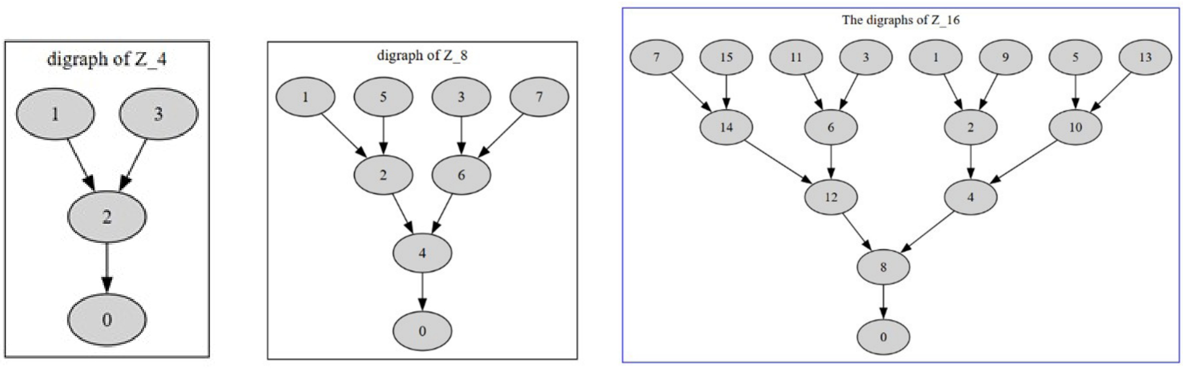


Figure 6: The diagraph of $\Gamma(\mathbb{Z}_4),\Gamma(\mathbb{Z}_8)$ and $\Gamma(\mathbb{Z}_{16})$

If we see that the digraphs are tree digraph and the distance of $\Gamma\left(\mathbb{Z}_{\left(4.2^{n-1}\right)}\right)$ is $d\left(\Gamma\left(\mathbb{Z}_{\left(4.2^{n-1}\right)}\right)\right)=2+$, $(n-1)$ and hence $d\left(\Gamma(\mathbb{Z}_4)\right)=2, d\left(\Gamma(\mathbb{Z}_8)\right)=3$, and $d\left(\Gamma(\mathbb{Z}_{16})\right)=4$.

Example 5.10. Let $G=\Gamma(\mathbb{Z}_{12})$ be a digraphs of the Q– algebra $(\mathbb{Z}_{12};-,0)$.

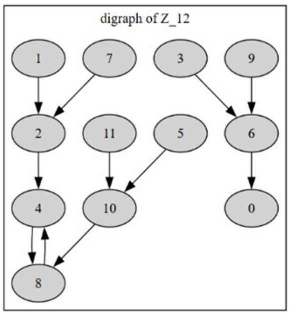


Figure 7: The diagraph of $\Gamma(\mathbb{Z}_{12})$

Then the digraph of G is disconnected and its vertex set splits into two connected components $V(G_1)$ = {0, 3, 6, 9} and $V(G_2)$ = {1, 2, 4, 5, 7, 8. 10, 11}. In particular, G_1 corresopnds to the set of multiples of 3 in $\mathbb{Z}_{12}$, and by Theorem 5.8 G_1 determines a sub Q– algebra of $(\mathbb{Z}_{12};-,0)$.

Theorem 5.11. *The Q– digraph of $\Gamma(\mathbb{Z}_n)$ is regular if and only if n is odd and connected.*

Proof. Suppose that Q– digraph of $\Gamma(\mathbb{Z}_n)$ is regular then out-degree = in-degree so by Remark 5.4. n is odd and connected. Conversely, Suppose that the Q– digraph of $\Gamma(\mathbb{Z}_n)$, n is odd and connected, then by Remark 5.4. for every $v\in\Gamma(\mathbb{Z}_n)$ the degree of vertex v is $d^-(v)=d^+(v)=1$. Thus the Q– digraph of $\Gamma(\mathbb{Z}_n)$ is regular. □

Remark 5.12. A Q– digraph of $\Gamma(\mathbb{Z}_{4.2^{n-1}})$ with 4.2^{n-1} vertices contains (4.2^{n-1})-1 directed edge.

Theorem 5.13. *The Q– digraph of $\Gamma(\mathbb{Z}_{4.2^{n-1}})$ is anti - arborescence.*

Proof. By Theorem 5.5. The Q– digraph of $\Gamma(\mathbb{Z}_{4.2^{n-1}})$ is in - tree and for every vertex $v\in Z_{4.2^{n-1}}$, there is exactly one unique directed path from v to the root 0. Thus the Q- digraph of $\Gamma(\mathbb{Z}_{4.2^{n-1}})$ is anti - arborescence. □

6. Discussion

The characterization of Von-Neumann regular elements in $\mathbb{Z}_n$ as well as the tree and anti-arborescence structure of the associated Q-digraphs provides insight into the algebraic and combinatorial interplay in Q-algebras. This highlights the potential to apply these methods to study regularity phenomena in more general algebraic systems. Our results focus primarily on the Q-algebra $\mathbb{Z}_n$ and its associated Q-digraphs. Extending these characterizations to infinite Q-algebras or other classes of non-commutative Q-algebras remains an open challenge. Future work could extend these constructions to other algebraic structures related to Q-algebras, such as BCK- or BCH-algebras, or explore applications in algebraic graph theory and categorical frameworks.

7. Conclusion

This paper extends Von-Neumann regularity concepts to Q-algebras, defining Q-digraphs capturing their algebraic structure. It characterizes regular elements in $\mathbb{Z}_n$ and reveals that Q-digraphs of $\mathbb{Z}_{4.2^{n-1}}$

form tree-like structures. This unifies and generalizes previous algebraic graph theories for BCK-, BCI-, and ring-based graphs. The results deepen connections between algebra and directed graph theory, opening paths for broader algebraic and combinatorial studies.

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