Results in Nonlinear Analysis 8 (2025) No. 2, 305–313
https://doi.org/10.31838/rna/2025.08.02.023
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Perturbed statistical convergence

Ceylan Yalçin

Department of Mathematics, Çankaya University, Ankara, Turkey

Abstract

This paper examines the basic features of perturbed statistical convergence in the context of perturbed metric spaces. The suggested method expands on the standard concept of statistical convergence by using a perturbation function that shows the errors that might happen while measuring distance. The relations of this new type of convergence with classical and statistical convergence are discussed in detail. There are some examples and counterexamples that support the new theoretical results.

Mathematics Subject Classification (2010): 40A35, 54E35

Key words and phrases: Statistical Convergence;Metric Spaces; Perturbed Metric Spaces.

1. Introduction

Mathematics contains numerous distinct sub-topics and although these topics might seem independent at first, they are often fundamentally interconnected. One of the main reasons why such fields are interconnected is because similar concepts and challenges might arise in different fields. Thus, methods developed to overcome obstacles in a specific field may result in expansions in another field. However, a multidisciplinary viewpoint is necessary when adapting new approaches from one field into a solution of a situation in another.

Standard distance measurement functions often fail to model real-life problems due to their failure to account for measurement errors. A newly introduced metric-like function that considers measurement errors was first proposed in metric fixed point theory [9].

This paper focuses on the possibility of applying the newly introduced perturbed metric functions in the context of statistical convergence, aiming to construct a more realistic and error-tolerant convergence structure.

Email address: cyalcin@cankaya.edu.tr (Ceylan Yalçin)

Received May 9, 2025; Accepted July 22, 2025; Online August 28, 2025

2. Preliminaries

Errors occur in almost all measurements. Consequently, there is always a chance that measurements may be wrong; such errors may be due to human factor, environmental influences, measurement sensitivity, or other causes. For instance, cables used in energy transmission face higher temperatures during the summer months owing to rising air temperatures. The rise in the cable's temperature results in an expansion of its length. Therefore, the length of the cable is *perturbed* depending on the season.

In order to integrate these measurement mistakes into mathematical models, "Perturbed metric space" structure is developed by Jleli and Samet in [9]. The idea behind the perturbed metric space is fundamentally logical. It is predicated on the notion that the "real distance" should be equivalent to the difference between the error and the measured distance. Thus, the notion of metric space moves one step closer to reality. Such generalizations of metric structures can also be placed within the broader framework of hyperconvex and abstract convex spaces studied in KKM theory; see [13].

The definition of perturbed metric space will now be presented.

Definition 2.1 *Let $D, P : X \times X \to [0,\infty)$ be two mappings. We say that D is a* ***perturbed metric*** *on X with respect to P if the function*

$$d(x,y) := D(x,y) - P(x,y)$$

defines a metric on X. That is, for all $x,y,z \in X$, the following conditions hold:

- $D(x,y) - P(x,y) \geq 0$,
- $D(x,y) - P(x,y) = 0$ if and only if $x = y$,
- $D(x,y) - P(x,y) = D(y,x) - P(y,x)$,
- $D(x,y) - P(x,y) \leq D(x,z) - P(x,z) + D(z,y) - P(z,y)$.

In this context, the function P is referred to as a **perturbation**, the difference $d = D - P$ is called the **exact metric**, and the triple (X, D, P) is termed a **perturbed metric space**.

Recent and ongoing studies have been conducted on this subject. (See also [2],[3], [10]).

Following this definition, a mathematician would undoubtedly want to see an example to demonstrate the accuracy of the definition. In [9], the reader can find several examples of perturbed metric spaces, illustrating their differences from regular metric spaces. Based on these examples, the following example with a similar structure is given.

Example 2.2 *Let $X = \mathbb{R}$ and define $P, D : X \times X \to [0,\infty)$ by*

$$P(x,y) = 1 + \sin(x+y), \qquad D(x,y) = |x-y| + P(x,y).$$

Then the *exact metric* is

$$d(x,y) = D(x,y) - P(x,y) = |x-y|.$$

Since d is the usual metric on $\mathbb{R}$, it satisfies non-negativity, identity of indiscernibles, symmetry and the triangle inequality. Therefore $(\mathbb{R}, D, P)$ is a **perturbed metric space** with perturbation P and exact metric d. Hence, D is not a metric, but can be viewed as a *perturbed metric*.

Consequently, D is not a metric, but it satisfies the conditions of a perturbed metric.

At the base of the surface, the oscillations are clearly aligned with the $x - y$ direction in the xy -plane, producing wave-like patterns parallel to the diagonal.

This article also includes the notion of statistical convergence as a primary topic. Classical convergence depends on each of the terms in a sequence behave, but statistical convergence allows for surprising collections of indexes with zero density.

We would like to emphasise the essential principles of this subject.

$$\delta(M) = \lim_{n\to\infty} \frac{\#\{m \leq n : m \in M\}}{n}$$

where $\#M$ denotes the *cardinality* of a subset $M \subset \mathbb{N}$.

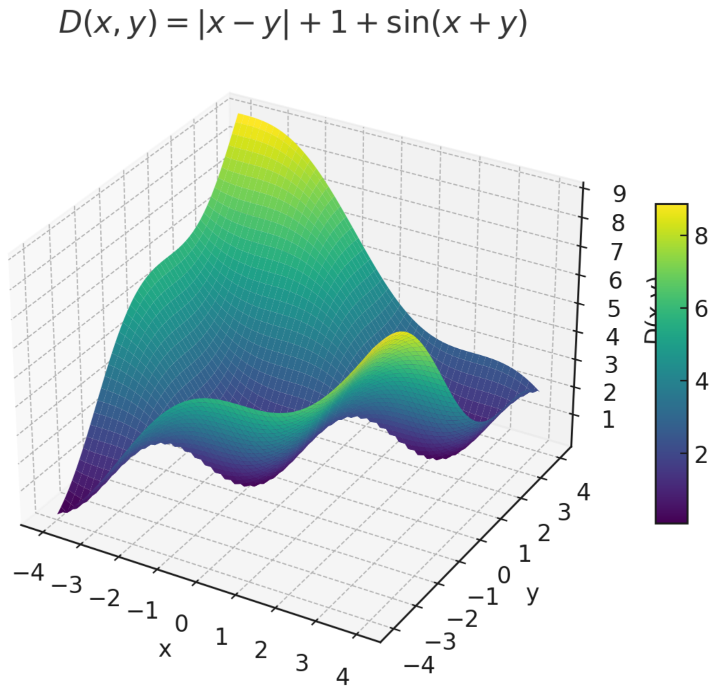

Figure 1: The graph of $D(x, y)$

The statistical convergence of a sequence is characterised using the concept of natural/asymptotic density. The following definition will clarify this connection.

Definition 2.3 *Let (x_k) be a sequence of real numbers. (x_k) is said to be statistically convergent to a number x if, for every $\varepsilon > 0$,*

$$\delta(\{m : |x_m - x| \geq \varepsilon\}) = 0 \tag{1}$$

Or, to put equation (1) another way

$$\lim_{n\to\infty} \frac{\#\{m \leq n : |x_m - x| \geq \varepsilon\}}{n} = 0, \tag{2}$$

holds. In the meantime, we use the next notation to denote this statistical limit:

$$st - \lim_{k\to\infty} x_k = x.$$

The statistical convergence of a sequence is characterised using the concept of natural/asymptotic density in [6] and [7].

It is well known that any convergent sequence statistically converges to same limit point. Nevertheless, the reverse is not generally true. This relaxation makes the convergence idea more flexible, which is notably helpful in analysis and summability theory.

Finally, we will emphasize the statistical version of the commonly recognized notion of the "Cauchy sequence".

Definition 2.4 *A sequence $x = (x_k)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that*

$$\delta(\{m : |x_m - x_{n_0}| \geq \varepsilon\}) = 0.$$

3. Main Results

Definition 3.1 *Let (X,D,P) be a perturbed metric space, and let (x_k) be a sequence in X. We say that (x_k) is **perturbed statistically convergent** to a point $x \in X$ if, for every $\varepsilon > 0$, the set*

$$\left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon + P(x_k, x)\right\}$$

has natural density zero, i.e.,

$$\delta\left(\left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon + P(x_k, x)\right\}\right) = 0. \tag{3}$$

In this case, we write

$$pst - \lim_{k \to \infty} x_k = x.$$

Remark 3.2 *We need to clarify that we used a different technique in Definition 3.1 from usual statistical convergence. In classical statistical convergence, it is assumed that a constant error or no error occurs while measuring the distance of the terms of a sequence from the number x. However, in this formulation, perturbation occurs while calculating the distance of each term of sequence to the point x. The process integrates a $P(x_k, x)$ perturbation amount, and permitting the convergence analysis's tolerance level to change dynamically in response to the index k.*

Remark 3.3 *Perturbed statistical convergence properly generalizes both classical statistical convergence and rough statistical convergence, depending on the choice of the perturbation function P:*

- If $P(x_k, x) = 0$ for all $k \in \mathbb{N}$, then the definition reduces to classical statistical convergence in [12].
- If $P(x_k, x) = r \geq 0$ is constant for all $k \in \mathbb{N}$, then the definition becomes:

$$\delta\left(\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon + r\}\right) = 0,$$

which corresponds to rough statistical convergence of degree r in [4].

Therefore, the proposed notion of convergence offers a unifying framework that includes both of these well-known types as special cases. In summary, PST-convergence acts as a unifying framework encompassing both statistical and rough statistical convergence, depending on the choice of the perturbation function.

Hence, the previous definition provides a more realistic framework since it works by including error terms.

This generalization aligns with earlier work on statistical convergence in alternative metric structures [1], [5], [8], [11], [14], [12].

Table 1 shows how the suggested PST-convergence fits in with other methods of convergence. In particular, it includes both statistical and rough statistical convergence as special cases and provides various methods to describe uncertainty in real-world measurements.

Table 1: Comparison of convergence methods

Type	Notation	Uses	Gen. of	Spec. case of
Classical	$\lim x_k = x$	Metric	—	All others (strict)
Statistical	$st - \lim x_k$	Density	Classical	Rough, PST
Rough Stat.	$st - \lim^r x_k$	Density + r	Statistical	PST with const. P
PST	$pst - \lim x_k$	Density + $P(x_k, x)$	Stat. + Rough	—

Example 3.4 *Let* $X = \mathbb{N}$. *Define the sequence* (x_k) *by*

$$x_k = \begin{cases} 1, & \text{if} k \in 2\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

This sequence is neither classically nor statistically convergent to 0, since

$$\delta\left(\{k \in \mathbb{N} : |x_k - 0| \geq \varepsilon\}\right) = \frac{1}{2} \neq 0.$$

Define a non-constant perturbation function:

$$P(x_k, 0) = \begin{cases} 1 - \dfrac{1}{\log(k+2)}, & \text{if } k \in 2\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Assuming the usual metric $D(x_k, 0) = |x_k - 0|$, we examine the inequality

$$D(x_k, 0) < P(x_k, 0) + \varepsilon.$$

For large values of k, this inequality is satisfied for all even indices, as $P(x_k, 0) + \varepsilon \to 1$, while $D(x_k, 0) = 1$ remains fixed. Therefore, the set

$$\left\{k \in \mathbb{N} : D(x_k, 0) \geq P(x_k, 0) + \varepsilon\right\}$$

is finite for any $\varepsilon > 0$, and has natural density zero.

To visualize this behavior, we fix $\varepsilon = 0.2$. The following graph clearly illustrates that for sufficiently large k, the inequality $D(x_k, 0) < P(x_k, 0) + \varepsilon$ holds.

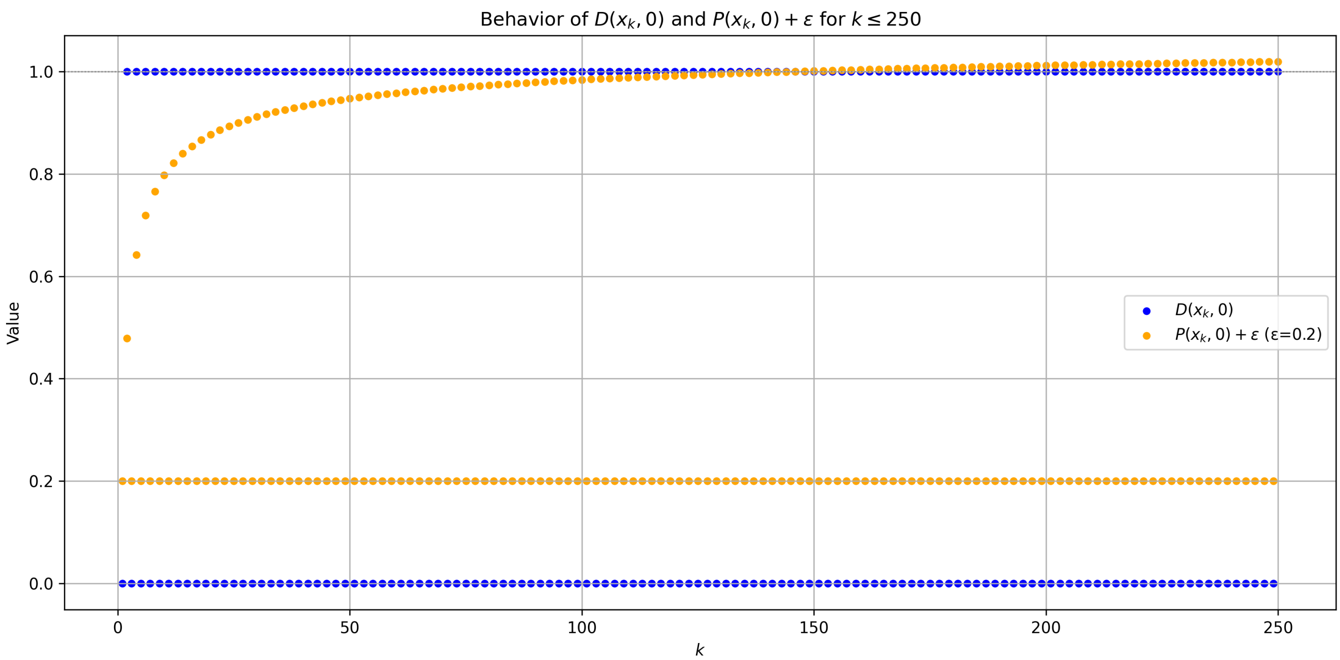

Figure 2: Graphical illustration of the inequality $D(x_k, 0) < P(x_k, 0) + \varepsilon$ for $\varepsilon = 0.2$. The plot shows that for large k, the perturbation term $P(x_k, 0) + \varepsilon$ becomes greater than $D(x_k, 0)$, which supports the PST-convergence of the sequence.

Theorem 3.5 *Let (X,D,P) be a perturbed metric space, and let $d = D - P$ be the exact metric on X. If a sequence (x_k) in X is convergent to a point $x \in X$ with respect to d, then (x_k) is perturbed statistically convergent to x.*

Proof. Let $\varepsilon > 0$ be given. Since (x_k) converges to x in the exact metric $d = D - P$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have

$$D(x_k, x) - P(x_k, x) < \varepsilon.$$

This implies

$$D(x_k, x) < P(x_k, x) + \varepsilon \quad \text{for all } k \geq N.$$

Therefore, the set

$$A := \{k \in \mathbb{N} : D(x_k, x) \geq P(x_k, x) + \varepsilon\}$$

is finite. Since every finite subset of $\mathbb{N}$ has natural density zero, we conclude that

$$\delta(A) = 0.$$

Hence, by Definition 3.1 , (x_k) is perturbed statistically convergent to x.

We will now investigate the relationship between the proposed convergence method and classical statistical convergence in order highlight the innovative aspect of our work:

Theorem 3.6 *Let (X,D,P) be a perturbed metric space and let (x_k) be a sequence in X such that*

$$\text{st} - \lim_{k \to \infty} x_k = x.$$

If the perturbation function $P(x_k, x)$ is bounded, i.e.,

$$\exists M > 0 \text{ such that } P(x_k, x) \leq M \text{ for all } k \in \mathbb{N},$$

then (x_k) is perturbed statistically convergent to x. That is,

$$\text{pst} - \lim_{k \to \infty} x_k = x.$$

Proof. Let $\varepsilon > 0$. Since (x_k) is statistically convergent to x, we have

$$\delta\left(\left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon\right\}\right) = 0.$$

Also, since $P(x_k, x) \leq M$, for all $k \in \mathbb{N}$, we know that

$$\varepsilon + P(x_k, x) \geq \varepsilon.$$

Hence, the set

$$\left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon + P(x_k, x)\right\} \subseteq \left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon\right\}$$

has natural density zero. Thus, by Definition 3.1, (x_k) is perturbed statistically convergent to x.

Remark 3.7 *The converse of Theorem 3.6 is not true in general. That is, there exist sequences which are perturbed statistically convergent but not statistically convergent. Such an example was given in Example 3.4*

The uniqueness of perturbed statistical limit is demonstrated by the following theorem.

Theorem 3.8 *Let* (x_k) *be a sequence in a perturbed metric space* (X,D,P). *If*

$$\text{pst}-\lim_{k\to\infty} x_k = x \quad \text{and} \quad \text{pst}-\lim_{k\to\infty} x_k = y,$$

then $x = y$.

Proof. Assume for contradiction that $x \neq y$. Since $d(x,y) = D(x,y) - P(x,y) > 0$, let $\varepsilon = \frac{1}{2}(D(x,y) - P(x,y)) > 0$.

Define the sets

$$A := \{k \in \mathbb{N} : D(x_k,x) \geq \varepsilon + P(x_k,x)\}, \quad B := \{k \in \mathbb{N} : D(x_k,y) \geq \varepsilon + P(x_k,y)\}.$$

Since x_k is perturbed statistically convergent to both x and y, we have

$$\delta(A) = 0 \quad \text{and} \quad \delta(B) = 0.$$

However, for all $k \in \mathbb{N}$, the triangle inequality implies

$$D(x,y) \leq D(x_k,x) + D(x_k,y),$$

so at least one of $D(x_k,x)$ or $D(x_k,y)$ must be greater than or equal to $\varepsilon + P(\cdot)$, which means $k \in A \cup B$ for all k.

Thus,

$$\delta(A \cup B) = 1 \leq \delta(A) + \delta(B),$$

which contradicts $\delta(A) = \delta(B) = 0$. Therefore, $x = y$.

The concept of a Cauchy sequence, common in classical analysis, will be redefined using a perturbation function. Consequently, a widely general Cauchy sequence structure with estimated error sensitivity will be developed.

Definition 3.9 *Let* (X,D,P) *be a perturbed metric space. A sequence* (x_k) *in* X *is called a perturbed statistically Cauchy sequence if, for every* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$\delta\left(\left\{k \in \mathbb{N} : D(x_k, x_{n_0}) \geq \varepsilon + P(x_k, x_{n_0})\right\}\right) = 0.$$

With the following remark, we will explain the relationship between the ideas of statistical convergence and statistical Cauchy sequence created using the perturbation function.

Remark 3.10 *Every perturbed statistically convergent sequence is also perturbed statistically Cauchy.*

Proof. Assume that $\text{pst}-\lim x_k = x$. Let $\varepsilon > 0$ be given.

Define the set

$$A := \left\{k \in \mathbb{N} : D(x_k,x) \geq \varepsilon + P(x_k,x)\right\}.$$

By Definition 3.1, we have

$$\delta(A) = 0 \quad \text{orequivalently} \quad \delta(A^c) = 1.$$

Take some $n_0 \in A^c$. Then

$$D(x_{n_0},x) < \frac{\varepsilon}{2} + P(x_{n_0},x).$$

Now consider

$$\left\{k \in \mathbb{N} : D(x_k, x) < \frac{\varepsilon}{2} + P(x_k, x)\right\} \subset \left\{k \in \mathbb{N} : D(x_k, x_{n_0}) < \varepsilon + P(x_k, x_{n_0})\right\}.$$

Let $k \in A^c$, i.e., $D(x_k, x) < \frac{\varepsilon}{2} + P(x_k, x)$. Then by part (iv) of Definition 2.1, which is the generalization of the triangle inequality in perturbed metric spaces, we have:

$$\begin{aligned} D(x_k, x_{n_0}) - P(x_k, x_{n_0}) &\leq D(x_k, x) - P(x_k, x) + D(x_{n_0}, x) - P(x_{n_0}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence,

$$D(x_k, x_{n_0}) < \varepsilon + P(x_k, x_{n_0}),$$

which implies that

$$k \in \left\{k \in \mathbb{N} : D(x_k, x_{n_0}) < \varepsilon + P(x_k, x_{n_0})\right\}.$$

Therefore, the inclusion

$$\left\{k \in \mathbb{N} : D(x_k, x) < \frac{\varepsilon}{2} + P(x_k, x)\right\} \subset \left\{k \in \mathbb{N} : D(x_k, x_{n_0}) < \varepsilon + P(x_k, x_{n_0})\right\}$$

implies

$$\delta\left(\left\{k \in \mathbb{N} : D(x_k, x_{n_0}) < \varepsilon + P(x_k, x_{n_0})\right\}\right) = 1.$$

Hence, we conclude that

$$\delta\left(\left\{k \in \mathbb{N} : D(x_k, x_{n_0}) \geq \varepsilon + P(x_k, x_{n_0})\right\}\right) = 0,$$

and by Definition 3.1, the sequence is perturbed statistically Cauchy.

Even if PST-convergence gives you more options, it turns out that most of the time, these sequences act like usual convergent ones. The next theorem makes this viewpoint more official.

Theorem 3.11 *Let (X, D, P) be a perturbed metric space. If a sequence (x_k) in X is perturbed statistically convergent to a point $x \in X$, then there exists a sequence (y_k) in X such that:*

1. $y_k = x_k$ for all k outside a set of natural density zero,
2. (y_k) converges to x with respect to the exact metric $d = D - P$.

Proof. Let $\varepsilon_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Since (x_k) is perturbed statistically convergent to x, we have

$$A_n := \left\{k \in \mathbb{N} : D(x_k, x) \geq \varepsilon_n + P(x_k, x)\right\}$$

satisfies $\delta(A_n) = 0$ for each n.

Define the exceptional set

$$A := \bigcup_{n=1}^{\infty} A_n.$$

Then $\delta(A) = 0$ by countable subadditivity of natural density.
Now define a new sequence (y_k) by

$$y_k = \begin{cases} x_k, & \text{if } k \notin A, \\ x, & \text{if } k \in A. \end{cases}$$

Clearly, $y_k = x_k$ for all $k \notin A$, and $y_k = x$ for $k \in A$.
Now observe that for $k \notin A_n$, we have

$$D(y_k, x) - P(y_k, x) = D(x_k, x) - P(x_k, x) < \frac{1}{n}.$$

For $k \in A$, $y_k = x$, so $D(y_k, x) - P(y_k, x) = 0$.
Hence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$d(y_k, x) = D(y_k, x) - P(y_k, x) < \varepsilon.$$

Therefore, $y_k \to x$ with respect to the exact metric $d = D - P$.

Remark 3.12 *This result emphasizes that perturbed statistical convergence implies convergence in the exact metric after modifying the sequence on a set of natural density zero.*

References

[1] R. Abazaria, Statistical convergence in G-metric spaces, Filomat 36(5) (2022) 1461–1468.
[2] G. Alsahli, E. Karapınar, P. Shahi, On rational contractions in perturbed metric spaces, Research in Mathematics 12(1) (2025) 1–10.
[3] G. Alsahli, P. Shahi, E. Karapınar, On perturbed-Stau-contractions, AIMS Mathematics 10(5) (2025) 11976–11985.
[4] S. Aytar, Rough Statistical Convergence, Numerical Functional Analysis and Optimization, 29(3–4)(2008) 291–303.
[5] B. Bilalov, T. Nazarova, On statistical convergence in metric spaces, J. Math. Res. 7(1) (2015) 37–43.
[6] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[7] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[8] M. İlkhan, E.E. Kara, On statistical convergence in quasi-metric spaces, Demonstr. Math. 52(1) (2019) 225–236.
[9] M. Jleli, B. Samet, On Banach's fixed point theorem in perturbed metric spaces, J. Appl. Anal. Comput. 15(2) (2025) 993–1001.
[10] E. Karapınar, C.M. Păcurar, P. Shahi, Interpolative contractions on perturbed metric spaces, An. Şt. Univ. Ovidius Constanța (to appear) (2025).
[11] K. Li, S. Lin, Y. Ge, On statistical convergence in cone metric spaces, Topol. Appl. 196 (2015) 641–651.
[12] F. Nuray, Statistical convergence in partial metric spaces, Korean J. Math. 30(1) (2022) 155–160.
[13] S. Park, Generalizations of hyperconvex metric spaces, Results in Nonlinear Analysis 2(2) (2019) 71–82.
[14] R. Sunar, Statistical convergence in A-metric spaces, Fundam. Contemp. Math. Sci. 5(2) (2024) 83–93.