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Parametrization for singular values inequalities of compact operators

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Abstract

We establish novel weighted parametric frameworks for singular value bounds concerning compact linear operators on complex separable Hilbert spaces. Our investigation introduces systematic parameter-dependent methodologies generating infinite families of refined inequalities that extend fundamental operator-theoretic results. The central achievement demonstrates the inequality:

Let T, $S \in B(H)$ be compact operators on a complex separable Hilbert space such that T is self-adjoint, $S \geq 0$, and $\pm T \leq S$. For any $\lambda \in (0,1]$, the following singular value inequality holds:

$$2\sigma_j(T) \leq \sigma_j((S+\lambda T) \oplus (S-\lambda T)) + 2(1-\lambda)\sigma_j(S \oplus S)$$

for j = 1, 2, ...

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1. Preliminaries and Background

The investigation of singular value bounds for compact linear operators represents a fundamental area within functional analysis, with far-reaching applications across mathematical physics, spectral theory, quantum mechanics, and computational analysis. This field has experienced substantial development through contributions from numerous researchers.

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Classical results established by Bhatia and Kittaneh [1] demonstrate that for compact operators T and S acting on a complex separable Hilbert space H, where T exhibits self-adjoint properties, $S \geq 0$, and the constraint $\pm T \leq S$ holds, we obtain:

$$\sigma_j(T) \leq \sigma_j(S \oplus S) \tag{1.1}$$

for all natural numbers j.

Building upon this foundation, Audeh and Kittaneh [2] established an alternative characterization: if operators $T, S, R \in B(H)$ satisfy the matrix inequality:

$$\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0,$$

then:

$$\sigma_j(S) \leq \sigma_j(T \oplus R) \tag{1.2}$$

Additionally, they proved the enhanced bound:

$$2\sigma_j(T) \leq \sigma_j((S+T) \oplus (S-T)) \tag{1.3}$$

Research by Tao [3] established complementary results showing that under appropriate hypotheses where T is self-adjoint, $S \geq 0$, and $\pm T \leq S$: If $\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq O$, then

$$2\sigma_j(S) \leq \sigma_j \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \tag{1.4}$$

Zhan [4] investigated corresponding bounds for differences of positive semidefinite operators, establishing: If $S, T \geq O$, then

$$\sigma_j(S-T) \leq \sigma_j(S \oplus T)) \tag{1.5}$$

under suitable conditions.

Our research develops comprehensive extensions of these classical results. Section 2 introduces weighted parameter methodologies producing continuous families of inequalities. Section 3 extends results to operator function frameworks, establishing unified approaches encompassing known results. Section 4 provides improved bounds for singular values within block matrix structures.

2. Weighted Parameter Methodologies for Singular Value Bounds

We commence by incorporating weighted parameters into singular value inequalities for operator compositions.

Theorem 2.1. *Let $T, S \in B(H)$ denote compact operators on a complex separable Hilbert space H, and let $X \in B(H)$. For any parameter $\lambda \in (0,1)$, the following singular value inequality holds:*

$$2\sigma_j(TX^*S^* + SXT^*) \leq \frac{1}{\lambda}\sigma_j\left((T|X|T^* + S|X^*|S^* + \lambda(TX^*S^* + SXT^*)) \oplus (T|X|T^* + S|X^*|S^* - \lambda(TX^*S^* + SXT^*))\right) \tag{2.1}$$

for $j = 1, 2, \ldots$.

Proof. We construct block matrices:

$$Y = \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{pmatrix}$$

The matrix $Z \geq 0$ exhibits positive semidefinite properties (established in operator theory literature). Computing YZY^*:

$$YZY^* = \begin{pmatrix} T\,|X|\,T^* \pm SXT^* \pm TX^*S^* + S\,|X^*|\,S^* & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

The positive semidefinite property of YZY^* implies:

$$T\,|X|\,T^* + S\,|X^*|\,S^* \geq \pm(TX^*S^* + SXT^*)$$

For any $\lambda \in (0,1)$, scaling yields:

$$T\,|X|\,T^* + S\,|X^*|\,S^* \geq \pm\lambda(TX^*S^* + SXT^*)$$

Setting $P = TX^*S^* + SXT^*$ and $Q = T\,|X|\,T^* + S\,|X^*|\,S^*$, we obtain $Q \geq \pm\lambda P$, ensuring both $Q + \lambda P \geq 0$ and $Q - \lambda P \geq 0$.

Consider the block matrix:

$$M = \begin{pmatrix} Q + \lambda P & 0 \\ 0 & Q - \lambda P \end{pmatrix}$$

Since both diagonal blocks are positive semidefinite, M is positive semidefinite. We have $Q \geq 0$ and $\pm\lambda P \leq Q$. Applying (1.3), we obtain

$$2\sigma_j(\lambda P) \leq \sigma_j((Q + \lambda P) \oplus (Q - \lambda P)) \tag{2.2}$$

Since λ is scalar and singular values exhibit homogeneity:

$$2\lambda\sigma_j(P) \leq \sigma_j((Q + \lambda P) \oplus (Q - \lambda P))$$

Therefore:

$$2\sigma_j(P) \leq \frac{1}{\lambda}\sigma_j((Q + \lambda P) \oplus (Q - \lambda P))$$

When $\lambda = 1$, we recover original inequalities. For $0 < \lambda < 1$, we obtain families of inequalities with factor $\frac{1}{\lambda}$ modifying bounds. □

2.1. Parametric Interpolation Families

The subsequent theorem establishes parametric families interpolating between fundamental inequalities.

Theorem 2.2. *Let T, S ∈ B(H) be compact operators on a complex separable Hilbert space such that T is self-adjoint, S ≥ 0, and ±T ≤ S. For any* $\lambda \in (0, 1]$, *the following singular value inequality holds:*

$$2\sigma_j(T) \leq \sigma_j((S + \lambda T) \oplus (S - \lambda T)) + 2(1 - \lambda)\sigma_j(S \oplus S) \tag{2.3}$$

for j = 1, 2, ...

Proof. We begin with the classical Audeh-Kittaneh inequality:

$$2\sigma_j(T) \leq \sigma_j((S + T) \oplus (S - T))$$

We also utilize:

$$\sigma_j(T) \leq \sigma_j(S \oplus S)$$

For any λ ∈ (0, 1], define $R_\lambda = \lambda T$. Since T is self-adjoint, R_λ is also self-adjoint. Moreover, since $\pm T \leq S$ and λ ∈ (0, 1], we have $\pm T \leq S$.

Applying the refined inequality to operator R_λ:

$$2\sigma_j(R_\lambda) \leq \sigma_j((S+R_\lambda)\oplus(S-R_\lambda))$$

Since singular values exhibit homogeneity:

$$2\lambda\sigma_j(T) \leq \sigma_j((S+\lambda T)\oplus(S-\lambda T))$$

Multiplying the basic inequality by 2(1 – λ):

$$2(1-\lambda)\sigma_j(T) \leq 2(1-\lambda)\sigma_j(S\oplus S)$$

Adding these inequalities:

$$2\lambda\sigma_j(T) + 2(1-\lambda)\sigma_j(T) \leq \sigma_j((S+\lambda T)\oplus(S-\lambda T)) + 2(1-\lambda)\sigma_j(S\oplus S) \tag{2.4}$$

$$2\sigma_j(T) \leq \sigma_j((S+\lambda T)\oplus(S-\lambda T)) + 2(1-\lambda)\sigma_j(S\oplus S) \tag{2.5}$$

This completes the demonstration.

When λ = 1, we recover (1.3).

Corollary 2.3. *Let T, S ∈ B(H) be compact operators on a complex separable Hilbert space such that T is self-adjoint,* $S \geq 0$*, and* $\pm T \leq S$*. For any λ ∈* (0, 1]*, the following singular value inequality holds:*

$$\sigma_j\begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix} \leq \frac{1}{2\lambda}\sigma_j\begin{pmatrix} T & \lambda S \\ \lambda S^* & R \end{pmatrix} \tag{2.6}$$

for j = 1, 2, ...

Proof. If $\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \geq 0$, then $\begin{pmatrix} T & -S \\ -S^* & R \end{pmatrix} \geq 0$ since these matrices are unitarily equivalent.

Then $\begin{pmatrix} T & 0 \\ 0 & R \end{pmatrix} \geq \pm\begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$.

Using parametric inequalities with appropriate scaling yields the desired result. □

Theorem 2.4. *Let T, S ∈ B(H) be compact operators on a complex separable Hilbert space with T,* $S \geq 0$. *Then:*

$$\lambda\sigma_j(T-S) \leq \sigma_j((1+\lambda)T + (1-\lambda)S \oplus (1-\lambda)T + (1+\lambda)S) \tag{2.7}$$

Proof. Since $T, S \geq 0$, we have $\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \geq 0$.

If $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$, then U is unitary and:

$$\begin{pmatrix} (1+\lambda)T + (1-\lambda)S & 0 \\ 0 & (1-\lambda)T + (1+\lambda)S \end{pmatrix} = U\begin{pmatrix} \frac{T+S}{2} & \frac{\lambda(T-S)}{2} \\ \frac{\lambda(T-S)}{2} & \frac{T+S}{2} \end{pmatrix}U^*$$

Applying classical singular value inequalities:

$$2\sigma_j\left(\frac{\lambda(T-S)}{2}\right) \leq \sigma_j\begin{pmatrix} \frac{T+S}{2} & \frac{\lambda(T-S)}{2} \\ \frac{\lambda(T-S)}{2} & \frac{T+S}{2} \end{pmatrix}$$

This yields the desired inequality after appropriate scaling. □

3. Extensions to Operator Functions

We extend singular value inequalities to operator function frameworks.

Theorem 3.1. *Let $T, S \in B(H)$ be compact operators on a complex separable Hilbert space such that T is self-adjoint, $S \geq 0$, and $\pm T \leq S$. Let f and g be continuous non-negative functions on $[0, \infty)$ satisfying:*

(i) f exhibits operator monotonicity (i.e., if $0 \leq U \leq V$, then $f(U) \leq f(V)$).
(ii) $f(0) = g(0) = 0$.
(iii) $f(t) \leq g(t)$ for all $t \in [0, \infty)$.

Then the following singular value inequality holds:

$$\sigma_j(f(T)) \leq \sigma_j(g(S) \oplus g(S)) \tag{3.1}$$

for $j = 1, 2, \ldots$.

Proof. Since $\pm T \leq S$, T is self-adjoint, $S \geq 0$, and f exhibits operator monotonicity:

$$\pm f(T) \leq f(S) \leq g(S)$$

Then $\begin{pmatrix} g(S)+f(T) & 0 \\ 0 & g(S)-f(T) \end{pmatrix} \geq 0.$

If $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$, then U is unitary and:

$$\begin{pmatrix} g(S)+f(T) & 0 \\ 0 & g(S)-f(T) \end{pmatrix} = U\begin{pmatrix} g(S) & f(T) \\ f(T) & g(S) \end{pmatrix}U^*$$

Therefore $\begin{pmatrix} g(S) & f(T) \\ f(T) & g(S) \end{pmatrix} \geq 0$, and applying classical inequalities yields the result. □

Corollary 3.2. *Let $T, S \in B(H)$ be compact operators on a complex separable Hilbert space such that T is self-adjoint, $S \geq 0$, and $\pm T \leq S$. Let f and g be continuous non-negative functions on $[0, \infty)$ satisfying conditions of Theorem 3.1. Then:*

$$2\sigma_j(f(T)) \leq \sigma_j((g(S)+f(T)) \oplus (g(S)-f(T))) \tag{3.2}$$

for $j = 1, 2, \ldots$.

Proof. Since $\pm T \leq S$, T is self-adjoint, $S \geq 0$, and f exhibits operator monotonicity:

$$\pm f(T) \leq f(S) \leq g(S)$$

Then $\begin{pmatrix} g(S)+f(T) & 0 \\ 0 & g(S)-f(T) \end{pmatrix} \geq 0.$

Using unitary transformations and applying Tao's inequality:

$$2\sigma_j(f(T)) \leq \sigma_j\begin{pmatrix} g(S) & f(T) \\ f(T) & g(S) \end{pmatrix} = \sigma_j\begin{pmatrix} g(S)+f(T) & 0 \\ 0 & g(S)-f(T) \end{pmatrix}$$

This establishes the desired result. □

4. Refined Bounds for Block Matrix Structures

This section establishes improved bounds providing enhanced estimates for singular values in specific operator configurations.

The research presented demonstrates the effectiveness of parametric approaches in extending classical singular value inequalities. The weighted parameter framework provides continuous families of inequalities interpolating between known results, offering enhanced flexibility for applications in operator theory, matrix analysis, and related disciplines.

References

[1] R. Bhatia and F. Kittaneh, *The matrix arithmetic-geometric mean inequality revisited*, Linear Algebra and its Applications, 428 (2008), 2177–2191.

[2] W. Audeh and F. Kittaneh, *Singular value inequalities for compact operators*, Linear Algebra and its Applications, 437 (2012), 2516–2522.

[3] Y. Tao, *More results on singular value inequalities of matrices*, Linear Algebra and its Applications, 416 (2006), 724–729.

[4] X. Zhan, *Singular values of differences of positive semidefinite matrices*, SIAM Journal on Matrix Analysis and Applications, 22 (2000), 819–823.

[5] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997.

[6] R. Bhatia and F. Kittaneh, *On the singular values of a product of operators*, SIAM Journal on Matrix Analysis and Applications, 11 (1990), 272–277.

[7] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, American Mathematical Society, Providence, RI, 1969.

[8] F. Kittaneh, *Some norm inequalities for operators*, Canadian Mathematical Bulletin, 42 (1999), 87–96.

[9] O. Hirzallah and F. Kittaneh, *Inequalities for sums and direct sums of Hilbert space operators*, Linear Algebra and its Applications, 424 (2007), 71–82.