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Coefficients bounds for certain families of q-holomorphic multivalent functions involving Srivastava-Hadi-Darus operator

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Abstract

In this paper, a new family of $\mathfrak{q}$-analogue starlike functions is proposed based on multivalent functions and a modified $\mathfrak{q}$-Bernardi integral operator. The introduction of quantitative calculus tools is an important step in constructing such families, as it allows the reformulation of the analytical and geometric properties of functions in a more general form, by replacing traditional derivatives and integrals. A number of essential holomorphic properties of the proposed family are analyzed, including the initial coefficient, the distortion and growth theorems, along with the study of extreme values and radius theorems. Finally, the definition of this family is extended to include the neighborhood of point ξ.

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1. Introduction and Background

Quantum calculus has found wide applications in number theory, combinatorics, orthogonal polynomials, and hypergeometric functions [4, 5, 10–12, 28, 32, 33, 36, 37]. Its role in geometric function theory was first formalized by Srivastava in 1989 [29], followed by the introduction of -starlike functions in 1990 [18]. Since then, various $\mathfrak{q}$-calculus operators have been developed, leading to new analytic function classes and results concerning their geometric features and coefficient estimates.

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Over time, numerous researchers have explored various families of analytic functions constructed via the $\mathfrak{q}$-Ruscheweyh operator. A recent contribution [9] discusses several applications of quantum calculus within geometric function theory (GFT), whereas a survey by Srivastava [30] outlines further advancements, including the development of multiple $\mathfrak{q}$-operators derived through techniques intrinsic to GFT. Motivated by these findings, the present study aims to propose a new application of the integral operator proposed in [31], which incorporates the $\mathfrak{q}$-Bernardi operator. This operator is employed to define a new family of holomorphic functions.

In this examination, the setting is established within the family of holomorphic functions defined on the unit disc $\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\}$, considered through the geometric framework determined by the starlikeness properties exhibited by certain families of holomorphic functions (see [7]).

Let $\mathcal{A}_{\mathfrak{p}}$ represent the family of normalized holomorphic functions given by:

$$h(\xi) := \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}} \xi^{\mathfrak{m}}, (\mathfrak{p} \in \mathbb{N}; \xi \in \mathbb{D}). \tag{1}$$

Let $\mathcal{S} \subset \mathcal{A}$ be the family of univalent functions in $\mathbb{D}$, and let $\mathcal{T}_{\mathfrak{p}}$ denote its subfamily of the function h with the following series:

$$h(\xi) := \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} |a_{\mathfrak{m}}| \xi^{\mathfrak{m}}, (\mathfrak{p} \in \mathbb{N}; \xi \in \mathbb{D}). \tag{2}$$

The family $\mathcal{S}^*(\chi)$ $(0 \leqq \chi < 1)$ consists of the functions $h \in \mathcal{A}$ for which

$$\Re\left(\frac{\xi h'(\xi)}{h(\xi)}\right) > \chi.$$

The standard family of starlike functions corresponds to $\chi = 0$, so that $\mathcal{S}^*(0) = \mathcal{S}^*$.

In GFT, convolution is an important concept, applied to a function h from (1) and a function $\hbar$ given by:

$$\hbar(\xi) := \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} c_{\mathfrak{m}} \xi^{\mathfrak{m}}.$$

The commonly employed convolution (or Hadamard) product is defined as:

$$(h * \hbar)(\xi) := \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}} c_{\mathfrak{m}} \xi^{\mathfrak{m}}.$$

In this work, the tools of quantitative calculus were used to enhance the course of study and achieve new results. The $\mathfrak{q}$-analysis, derived by Jackson [19, 20], are important tools with multiple applications in many mathematical areas. The basic principles of $\mathfrak{q}$-analysis and how they can be employed within the analytical framework used in this work will be explained below.

The $\mathfrak{q}$-derivative operator $\partial_{\mathfrak{q}} : \mathcal{A}_{\mathfrak{p}} \to \mathcal{A}_{\mathfrak{p}}$, is applied as follows:

$$\partial_{\mathfrak{q}} h(\xi) := \begin{cases} \dfrac{h(\xi) - h(\mathfrak{q}\xi)}{(1-\mathfrak{q})\xi} & \xi \neq 0 \\ h'(0) & \xi = 0 \end{cases}, (\mathfrak{q} \in (0,1)). \tag{3}$$

It is readily to see for k, ξ in (1) that

$$\partial_{\mathfrak{q}} \left\{ \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}} \xi^{\mathfrak{m}} \right\} = [\mathfrak{p}]_{\mathfrak{q}} \xi^{\mathfrak{p}-1} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_{\mathfrak{q}} a_{\mathfrak{m}} \xi^{\mathfrak{m}-1}$$

with the number $[\mathfrak{m}]_{\mathfrak{q}} = \dfrac{1-\mathfrak{q}^k}{1-\mathfrak{q}}$ and $[0]_{\mathfrak{q}} = 0$.

The $\mathfrak{q}$-derivative satisfies the following essential rules:

$$\partial_{\mathfrak{q}}(uh(\xi) \pm vt(\xi)) = u\partial_{\mathfrak{q}}h(\xi) \pm v\partial_{\mathfrak{q}}t(\xi),$$
$$\partial_{\mathfrak{q}}(h(\xi)t(\xi)) = h(\mathfrak{q}\xi)\partial_{\mathfrak{q}}(t(\xi)) + t(\xi)\partial_{\mathfrak{q}}(h(\xi)),$$
$$\partial_{\mathfrak{q}}\left(\frac{h(\xi)}{t(\xi)}\right) = \frac{\partial_{\mathfrak{q}}(h(\xi))t(\xi) - h(\xi)\partial_{\mathfrak{q}}(t(\xi))}{t(\mathfrak{q}\xi)t(\xi)}, t(\mathfrak{q}\xi)t(\xi) \neq 0,$$
$$\partial_{\mathfrak{q}}(\log h(\xi)) = \frac{\ln \mathfrak{q}}{\mathfrak{q}-1}\frac{\partial_{\mathfrak{q}}(h(\xi))}{h(\xi)}$$

with $h, t \in \mathcal{A}_{\mathfrak{p}}$ and $u, v \in \mathbb{C}$ or $\mathbb{R}$.

The central theme of q-calculus lies in the study of q-analogs, inspired by the inherent symmetry of quantum calculus. Building on the preceding results, the present work aims to provide a further employment of the integral operator, namely Srivastava–Hadi–Darus operator, as established in [31]. This operator is employed to propose a family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$ of starlike functions. The operator is recalled below.

Srivastava-Hadi-Darus [31] recently provided a new modified of $\mathfrak{q}$-Bernardi operator for multivalent functions as follows:

$$\mathcal{B}^{\mathfrak{p}}_{s,\mathfrak{q}}h(\xi) := \begin{cases} \mathcal{B}^{\mathfrak{p}}_{1,\mathfrak{q}}\left(\mathcal{B}^{\mathfrak{p}}_{s-1,\mathfrak{q}}h(\xi)\right), & (s \in \mathbb{N}) \\ h(\xi), & (s = 0) \end{cases} = \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s} a_{\mathfrak{m}}\xi^{\mathfrak{m}}, \tag{4}$$

where

$$\mathcal{B}^{\mathfrak{p}}_{1,\mathfrak{q}}h(\xi) = \frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{\xi^{\varepsilon}}\int_0^{\xi} w^{\varepsilon-1}h(w)d_{\mathfrak{q}}w = \xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}a_{\mathfrak{m}}\xi^{\mathfrak{m}}, \quad (\varepsilon > -\mathfrak{p}).$$

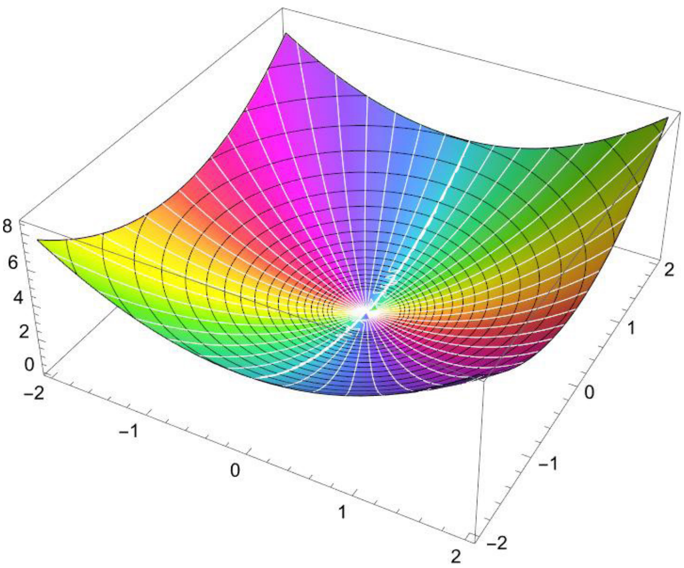

Figure 1: Plot of the operator (1) with $\mathfrak{p} = 2$, $\mathfrak{m} = 3$, $s = 5$, $\mathfrak{q} \to 1^-$, and $\varepsilon = -1$.

This operator represents a generalized formula for a number of familiar integral operators, such as Jung-Kim-Srivastava operator [8], $\mathfrak{q}$-Libra operator [24], $\mathfrak{q}$-Bernardi operator [24], ($\mathfrak{q}$, q)-Bernardi operator [23], and $\mathfrak{q}$-Srivastava-Attiya operator [27, 34].

This operator also has contributed to directing researchers attention to numerous studies focusing on the geometric properties of holomorphic functions related to $\mathfrak{q}$-calculus (e.g. [14–16, 21]).

Employing the function h in (2), the operator (4) can be formulated as:

$$\mathcal{B}_{s,\mathfrak{q}}^{\mathfrak{p}} h(\xi) := \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}} \right)^{s} a_{\mathfrak{m}} \xi^{\mathfrak{m}}.$$

The families considered in this study, constructed via the operator introduced in (4), are presented below.

Definition 1.1. *A holomorphic function $h(\xi) \in \mathcal{T}_{\mathfrak{p}}$ is considered to belong to the family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$ if and only if*

$$\left| \frac{\partial_{\mathfrak{q}} \mathcal{B}_{s,\mathfrak{q}}^{\mathfrak{p}} h(\xi) - [\mathfrak{p}]_{\mathfrak{q}} \xi^{\mathfrak{p}-1}}{\chi_1 \partial_{\mathfrak{q}} \mathcal{B}_{s,\mathfrak{q}}^{\mathfrak{p}} h(\xi) + \chi_2 [\mathfrak{p}]_{\mathfrak{q}} \xi^{\mathfrak{p}-1}} \right| \leqq \gamma, \tag{5}$$

where $0 \leqq \chi_1 \leqq 1, 0 < \chi_2 \leqq 1$ *and* $0 < \gamma \leqq 1$.

Remark 1.2. *By assigning values to parameters, several families are derived by Kim and Lee* [22] *(also* [1, 2, 17, 26]*), as follows:*

1. *For* $\mathcal{H}(\chi_1, \chi_2, \gamma, 0, 1, \mathfrak{q} \to 1-)$, *this family implies to*

$$\left| \frac{h'(\xi) - 1}{\chi_1 h'(\xi) + \chi_2} \right| \leqq \gamma.$$

2. *For* $\mathcal{H}(\chi_1, 1, \frac{1-\chi_1}{1+\chi_1^2}, 0, 1, \mathfrak{q} \to 1-)$, *this family implies to*

$$\Re(h'(\xi)) \geqq \chi_1.$$

3. *For* $\mathcal{H}(0, 1, \gamma, 0, 1, \mathfrak{q} \to 1-)$, *this family implies to*

$$| h'(\xi) - 1 | \leqq \gamma.$$

4. *For* $\mathcal{H}(1, 1, \gamma, 0, 1, \mathfrak{q} \to 1-)$, *this family implies to*

$$\left| \frac{h'(\xi) - 1}{h'(\xi) + 1} \right| \leqq \gamma.$$

Definition 1.3. *The function $h \in \mathcal{T}_{\mathfrak{p}}$ is a member of the family $\mathcal{H}_{\lambda_0}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$, $(0 \leqq \lambda_0 < 1)$ provided that there exists $t(\xi) \in \mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$ with*

$$\left| \frac{h(\xi)}{t(\xi)} - 1 \right| < 1 - \lambda_0, \ (\xi \in \mathbb{D}).$$

Definition 1.4. [3, 13, 35] *The λ-neighborhood of $h \in \mathcal{T}_{\mathfrak{p}}$, for $\lambda \geqq 0$, is specified as:*

$$\mathcal{N}_{\mathfrak{p}}^{\mathfrak{q}}(h, t) := \left\{ t : t = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} c_{\mathfrak{m}} \xi^{\mathfrak{m}} \in \mathcal{T}_p \text{ and } \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \mathfrak{m} \left| a_{\mathfrak{m}} - c_{\mathfrak{m}} \right| \leqq \lambda \right\}. \tag{6}$$

Taking $h \in \mathcal{T}_{\mathfrak{p}}$ with $h(\xi) = \xi^{\mathfrak{p}}$, the desired statement follows directly

$$\mathcal{N}_{\mathfrak{p}}^{\mathfrak{q}}(h, t) := \left\{ t : t = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} c_{\mathfrak{m}} \xi^{\mathfrak{m}} \in \mathcal{T}_p \text{ and } \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \mathfrak{m} \left| c_{\mathfrak{m}} \right| \leqq \lambda \right\}. \tag{7}$$

This investigation addresses some well-known geometric properties. These properties include initial coefficient, the distortion and growth theorems, extreme points, and radius theory, employing techniques associated with multivalent functions.

2. Essential Findings

Throughout this work, the estimates of the parameters will be $0 \leqq \chi_1 \leqq 1$, $0 < \chi_2 \leqq 1$, $0 < \gamma \leqq 1$, and $s \in \mathbb{N}_0$.

Theorem 2.1. *A function $h(\xi) \in \mathcal{T}_\mathfrak{p}$ is included in the family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$ if and only if*

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} (\chi_1 \gamma + 1) \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{m}+\varepsilon]_\mathfrak{q}} \right)^s |a_\mathfrak{m}| \leqq \gamma [\mathfrak{p}]_\mathfrak{q} (\chi_1 + \chi_2). \tag{8}$$

The finding is also sharp for the function expressed as

$$h(\xi) = \xi^\mathfrak{p} - \frac{\gamma [\mathfrak{p}]_\mathfrak{q} (\chi_1 + \chi_2)}{[\mathfrak{m}]_\mathfrak{q} (\chi_1 \gamma + 1) \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{m}+\varepsilon]_\mathfrak{q}} \right)^s} a_\mathfrak{m} \xi^\mathfrak{m}, \; (\mathfrak{m} \geqq \mathfrak{p}+1). \tag{9}$$

Proof. Given that h is part of $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$, we can derive from relation (1) that

$$\begin{aligned} \left| \frac{\partial_\mathfrak{q} \mathcal{B}_{s,\mathfrak{q}}^\mathfrak{p} h(\xi) - [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}}{\chi_1 \partial_\mathfrak{q} \mathcal{B}_{s,\mathfrak{q}}^\mathfrak{p} h(\xi) + \chi_2 [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}} \right| &= \left| \frac{[\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1} - [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}}{\chi_1 \left([\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1} \right) + \chi_2 [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}} \right| \\ &= \left| \frac{\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1}}{\chi_1 \left([\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1} \right) + \chi_2 [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}} \right| \\ &\leqq \gamma. \end{aligned}$$

It is a well-established fact that $\Re(\xi) \leqq |\xi|$; consequently, we obtain:

$$\Re \left\{ \frac{\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1}}{\chi_1 \left([\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \xi^{\mathfrak{m}-1} \right) + \chi_2 [\mathfrak{p}]_\mathfrak{q} \xi^{\mathfrak{p}-1}} \right\} \leqq \gamma.$$

By selecting a real ξ and allowing $\xi \to 1-$, we get

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \leqq \gamma \left[\chi_1 \left([\mathfrak{p}]_\mathfrak{q} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_\mathfrak{q} \left(\frac{[\mathfrak{p}+\varepsilon]_\mathfrak{q}}{[\mathfrak{p}+\varepsilon]_\mathfrak{q}} \right)^s a_\mathfrak{m} \right) + \chi_2 [\mathfrak{p}]_\mathfrak{q} \right]$$

which is exactly the emphasis (8).

Conversely, assume that inequality (8) is valid, and let $\xi \in \partial\mathbb{D}$. From (5), it follows that:

$$\left|\partial_{\mathfrak{q}}\mathcal{B}^{\mathfrak{p}}_{s,\mathfrak{q}}h(\xi) - [\mathfrak{p}]_{\mathfrak{q}}\xi^{\mathfrak{p}-1}\right| - \gamma\left|\chi_1\partial_{\mathfrak{q}}\mathcal{B}^{\mathfrak{p}}_{s,\mathfrak{q}}h(\xi) + \chi_2[\mathfrak{p}]_{\mathfrak{q}}\xi^{\mathfrak{p}-1}\right|$$

$$\leqq \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}[\mathfrak{m}]_{\mathfrak{q}}\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s} a_{\mathfrak{m}}\left|\xi\right|^{\mathfrak{m}-1} - \gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2) + \gamma\chi_1\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}[\mathfrak{m}]_{\mathfrak{q}}\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s} a_{\mathfrak{m}}\left|\xi\right|^{\mathfrak{m}-1}$$

$$\leqq \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}[\mathfrak{m}]_{\mathfrak{q}}(\gamma\chi_1+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s} a_{\mathfrak{m}} - \gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2) \leqq 0.$$

As stated by the maximum modulus theorem, $h\xi$ is included in $\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$. □

Theorem 2.2. *For* $h(\xi)\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$, *we deduce that*

$$|\xi|^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}|\xi|^{\mathfrak{p}+1} \leqq |h(\xi)|$$

$$\leqq |\xi|^{\mathfrak{p}} + \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}|\xi|^{\mathfrak{p}+1}.$$

Proof. If $h(\xi)\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$. The upper constraint of $h\xi$ is stated by

$$|h(\xi)| \leqq |\xi|^{\mathfrak{p}} + |\xi|^{\mathfrak{p}+1}\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}a_{\mathfrak{m}}, \tag{10}$$

and Theorem 2.1 gives

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}a_{\mathfrak{m}} \leqq \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}. \tag{11}$$

Employing (11) in (10) yields

$$|h(\xi)| \leqq |\xi|^{\mathfrak{p}} + \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}|\xi|^{\mathfrak{p}+1}, \tag{12}$$

and also the inequality (11) gives

$$|h(\xi)| \geqq |\xi|^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}|\xi|^{\mathfrak{p}+1}. \tag{13}$$

So, by combining (12) with (13) we arrive at the target result. □

Theorem 2.3. *For* $h(\xi)\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$, *we get*

$$[\mathfrak{p}]_{\mathfrak{q}}|\xi|^{\mathfrak{p}-1} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}|\xi|^{\mathfrak{p}} \leqq |\partial_{\mathfrak{q}}h(\xi)|$$

$$\leqq [\mathfrak{p}]_{\mathfrak{q}} |\xi|^{\mathfrak{p}-1} + \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^s} |\xi|^{\mathfrak{p}}.$$

Proof. If $h(\xi) \in \mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$. The upper constraint of $\partial_{\mathfrak{q}} h(\xi)$ is stated by

$$|\partial_{\mathfrak{q}} h(\xi)| \leqq [\mathfrak{p}]_{\mathfrak{q}} |\xi|^{\mathfrak{p}-1} + [\mathfrak{p}+1]_{\mathfrak{q}} |\xi|^{\mathfrak{p}} \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}}.$$

Substituting the value of (11) and using analogue technique as before, we arrive at the desired boundaries. □

Theorem 2.4. *If the functions h and $\hbar$ are in the family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$, where*

$$h(\xi) = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}} \xi^{\mathfrak{m}},$$

$$\hbar(\xi) = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} b_{\mathfrak{m}} \xi^{\mathfrak{m}}.$$

Then $\mathcal{F}$ is expressed as

$$\mathcal{F} = (1-\omega)h(\xi) + \omega\hbar(\xi) = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} d_{\mathfrak{m}} \xi^{\mathfrak{m}},\ (0 \leqq \omega \leqq 1;\ d_{\mathfrak{m}} = (1-\omega)a_{\mathfrak{m}} + \omega b_{\mathfrak{m}})$$

is also within the family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$.

Proof. If both functions h and $\hbar$ is included in the family $\mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$, then using inequality (11) leads to:

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_{\mathfrak{q}} (\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s |b_{\mathfrak{m}}|$$
$$= (1-\omega)\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_{\mathfrak{q}} (\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s |a_{\mathfrak{m}}|$$
$$+ \omega\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_{\mathfrak{q}} (\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s |b_{\mathfrak{m}}|$$
$$\leqq (1-\omega)\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2) + \omega\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2) = \gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)$$

which satisfies the demonstration. □

Theorem 2.5. *If $h_{\mathfrak{p}}(\xi) = \xi^{\mathfrak{p}}$,*

$$h_{\mathfrak{m}}(\xi) = \xi^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s} \xi^{\mathfrak{m}},\ (\mathfrak{m} \geqq \mathfrak{p}+1),$$

then $h \in \mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$, if and only if h is formulated by

$$h(\xi) = \omega_{\mathfrak{p}} h_{\mathfrak{p}}(\xi) + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} h_{\mathfrak{m}}(\xi), \tag{14}$$

with $\omega_{\mathfrak{m}} \geqq 0$ *and* $\omega_p + \sum_{\mathfrak{m}=\mathfrak{p}+}^{\infty} \omega_{\mathfrak{m}} = 1$.

Proof. Define

$$h(\xi) = \omega_{\mathfrak{p}} h_{\mathfrak{p}}(\xi) + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} h_{\mathfrak{m}}(\xi),$$

therefore

$$h(\xi) = \left(1 - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}}\right)\xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} \left\{ \xi^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s} \xi^{\mathfrak{m}} \right\}$$

$$= \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} \left\{ \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s} \xi^{\mathfrak{m}} \right\}.$$

Then,

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} [\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s \omega_{\mathfrak{m}} \cdot \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}$$

$$= \gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2) \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} = (1 - \omega_{\mathfrak{p}})\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2) \leqq \gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2).$$

This confirms that h fulfills condition (14), and thus $h \in \mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$.

Now, assume conversely that $h \in \mathcal{H}(\chi_1, \chi_2, \gamma, s, \mathfrak{p}, \mathfrak{q})$. As

$$a_{\mathfrak{m}} \leqq \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}, \qquad (\mathfrak{m} \geqq \mathfrak{p}+1).$$

Setting

$$\omega_{\mathfrak{m}} = \frac{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma + 1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1 + \chi_2)} a_{\mathfrak{m}}.$$

Thus,

$$h(\xi) = \xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}} \xi^{\mathfrak{m}}$$

$$= (\omega_{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}})\xi^{\mathfrak{p}} - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}\xi^{\mathfrak{m}}$$

$$= \omega_{\mathfrak{p}}\xi^{\mathfrak{p}} + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} \left(\xi^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}\xi^{\mathfrak{m}} \right)$$

$$= \omega_{\mathfrak{p}} h_{\mathfrak{p}}(\xi) + \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \omega_{\mathfrak{m}} h_{\mathfrak{m}}(\xi),$$

which satisfies the demonstration. □

Corollary 2.6. *The extreme values of* $\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ *are indicated by*

$$h_{\mathfrak{p}}(\xi) = \xi^{\mathfrak{p}},\, h_{\mathfrak{m}}(\xi) = \xi^{\mathfrak{p}} - \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s}\xi^{\mathfrak{m}},\, (\mathfrak{m} \geqq \mathfrak{p}+1).$$

3. Radius theorems

This section examines the radius problems for functions that holomorphic families of multivalent functions. Utilizing the $\mathfrak{q}$-analysis, we provide the following families of order ϖ (refer to [6]).

$$\mathcal{TS}_{\mathfrak{q}}^{\mathfrak{p},*}(\varpi) := \left\{ h : h \in \mathcal{T}_{\mathfrak{p}} \text{ and } \Re\left(\frac{\xi\partial_{\mathfrak{q}}h(\xi)}{h(\xi)}\right) > \varpi \quad (0 \leqq \varpi < \mathfrak{p}; \xi \in \mathbb{D}) \right\},$$

$$\mathcal{TC}_{\mathfrak{q}}^{\mathfrak{p}}(\varpi) := \left\{ h : h \in \mathcal{T}_{\mathfrak{p}} \text{ and } \Re\left(\frac{\partial_{\mathfrak{q}}\left(\xi\partial_{\mathfrak{q}}h(\xi)\right)}{\partial_{\mathfrak{q}}(h(\xi))}\right) > \varpi \quad (0 \leqq \varpi < \mathfrak{p}; \xi \in \mathbb{D}) \right\}, \tag{15}$$

$$\mathcal{TK}_{\mathfrak{q}}^{\mathfrak{p}}(\varpi) := \left\{ h : h \in \mathcal{T}_{\mathfrak{p}} \text{ and } \Re\left(\frac{\partial_{\mathfrak{q}}h(\xi)}{\xi^{\mathfrak{p}-1}}\right) > \varpi \quad (0 \leqq \varpi < \mathfrak{p}; \xi \in \mathbb{D}) \right\}.$$

Theorem 3.1. *For* $h \in \mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ *the multivalent function h is* ϖ*-starlike in* $|\xi| < \mathfrak{r}_1(\chi_1,\chi_2,\gamma,s,\varpi)$, *that is*

$$\mathfrak{r}_1(\chi_1,\chi_2,\gamma,s,\varpi) = \inf_{\mathfrak{m}} \left\{ \frac{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^s([\mathfrak{p}]_{\mathfrak{q}}-\varpi)}{([\mathfrak{m}]_{\mathfrak{q}}-\varpi)\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)} \right\}^{\frac{1}{e}},\ (e = \mathfrak{m}-\mathfrak{p};\, \mathfrak{m} \geqq \mathfrak{p}+1).$$

Proof. To complete the proof, it suffices to emphasize that

$$\left| \frac{\partial_{\mathfrak{q}}h(\xi)}{h(\xi)} - [\mathfrak{p}]_{\mathfrak{q}} \right| \leqq [\mathfrak{p}]_{\mathfrak{q}} - \varpi.$$

The computation of this family leads directly to

$$\left| \frac{\partial_{\mathfrak{q}}h(\xi)}{h(\xi)} - [\mathfrak{p}]_{\mathfrak{q}} \right| \leqq \frac{\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}([\mathfrak{m}]_{\mathfrak{q}} - [\mathfrak{p}]_{\mathfrak{q}})a_{\mathfrak{m}}|\xi|^{\mathfrak{m}-\mathfrak{p}}}{1 - \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} a_{\mathfrak{m}}|\xi|^{\mathfrak{m}-\mathfrak{p}}} \leqq [\mathfrak{p}]_{\mathfrak{q}} - \varpi. \tag{16}$$

Applying Theorem 2.1 leads to the inequality $a_{\mathfrak{m}}$ in (8).

The truth (16) is confirmed if

$$\frac{([\mathfrak{m}]_{\mathfrak{q}}-\varpi)\,|\xi|^{\mathfrak{m}-\mathfrak{p}}}{[\mathfrak{p}]_{\mathfrak{q}}-\varpi} \leqq \frac{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}.$$

The final simplification result is

$$|\xi| \leqq \left\{\frac{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}([\mathfrak{p}]_{\mathfrak{q}}-\varpi)}{([\mathfrak{m}]_{\mathfrak{q}}-\varpi)\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}\right\}^{\frac{1}{e}}, (e=\mathfrak{m}-\mathfrak{p}).$$

□

Theorem 3.2. *For $h\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ the multivalent function h is ϖ-close-to-convex in $|\xi|<\mathfrak{r}_2(\chi_1,\chi_2,\gamma,s,\varpi)$, that is*

$$\mathfrak{r}_2(\chi_1,\chi_2,\gamma,s,\varpi)=\inf_{\mathfrak{m}}\left\{\frac{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}([\mathfrak{p}]_{\mathfrak{q}}-\varpi)}{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}\right\}^{\frac{1}{e}}, (e=\mathfrak{m}-\mathfrak{p};\, \mathfrak{m}\geqq\mathfrak{p}+1).$$

Proof. To complete the proof, it suffices to emphasize that

$$\left|\frac{\partial_{\mathfrak{q}}h(\xi)}{\xi^{\mathfrak{p}-1}}-[\mathfrak{p}]_{\mathfrak{q}}\right| \leqq [\mathfrak{p}]_{\mathfrak{q}}-\varpi.$$

The computation of this family leads directly to

$$\left|\frac{\partial_{\mathfrak{q}}h(\xi)}{\xi^{\mathfrak{p}-1}}-[\mathfrak{p}]_{\mathfrak{q}}\right| \leqq \sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}[\mathfrak{m}]_{\mathfrak{q}}a_{\mathfrak{m}}\,|\xi|^{\mathfrak{m}-\mathfrak{p}} \leqq [\mathfrak{p}]_{\mathfrak{q}}-\varpi,$$

yields

$$\frac{\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}[\mathfrak{m}]_{\mathfrak{q}}a_{\mathfrak{m}}\,|\xi|^{\mathfrak{m}-\mathfrak{p}}}{[\mathfrak{p}]_{\mathfrak{q}}-\varpi} \leqq 1. \quad (17)$$

Applying Theorem 2.1 leads to the inequality $a_{\mathfrak{m}}$ in (8).

The truth (17) is confirmed if

$$\frac{[\mathfrak{m}]_{\mathfrak{q}}\,|\xi|^{\mathfrak{m}-\mathfrak{p}}}{[\mathfrak{p}]_{\mathfrak{q}}-\varpi} \leqq \frac{[\mathfrak{m}]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}.$$

The final simplification result is

$$|\xi| \leqq \left\{ \frac{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}([\mathfrak{p}]_{\mathfrak{q}}-\varpi)}{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)} \right\}^{\frac{1}{e}}, \ (e=\mathfrak{m}-\mathfrak{p}).$$

□

Theorem 3.3. *For* $h\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ *the multivalent function* h *is* ϖ*-convex in* $|\xi|<\mathfrak{r}_3(\chi_1,\chi_2,\gamma,s,\varpi)$, *that is*

$$\mathfrak{r}_3(\chi_1,\chi_2,\gamma,s,\varpi)=\inf_{\mathfrak{m}}\left\{ \frac{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[\mathfrak{m}+\varepsilon]_{\mathfrak{q}}}\right)^{s}([\mathfrak{p}]_{\mathfrak{q}}-\varpi)}{([\mathfrak{m}]_{\mathfrak{q}}-\varpi)\gamma(\chi_1+\chi_2)} \right\}^{\frac{1}{e}}, (e=\mathfrak{m}-\mathfrak{p};\mathfrak{m}\geqq\mathfrak{p}+1).$$

Proof. The desired result can be obtained by proving

$$\left|1+\frac{\xi\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}h(\xi))}{\partial_{\mathfrak{q}}h(\xi)}-[\mathfrak{p}]_{\mathfrak{q}}\right|\leqq[\mathfrak{p}]_{\mathfrak{q}}-\varpi$$

and following the method of reasoning in Theorem 3.1. □

4. λ-Neighborhood

Theorem 4.1. *Setting*

$$\lambda:=\frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}, \tag{18}$$

then $\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})\subset\mathcal{N}_{\mathfrak{p}}^{\mathfrak{q}}(h,t)$.

Proof. If the function $h(\xi)\in\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ has the form (2), then Theorem 2.1 generates

$$[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}|a_{\mathfrak{m}}|\leqq\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2),$$

thus,

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}a_{\mathfrak{m}}\leqq\frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}. \tag{19}$$

On contrary, we deduce from inequality (8) that

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}ka_{\mathfrak{m}}\leqq\frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}, \tag{20}$$

therefore

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} k a_{\mathfrak{m}} \leqq \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}} := \lambda, \tag{21}$$

which gives the main demonstrate. □

Theorem 4.2. *If* $t(\xi)\in \mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ *and*

$$\lambda_0 = 1-\frac{\lambda}{[\mathfrak{p}+1]_{\mathfrak{q}}}\cdot\frac{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}-\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}, \tag{22}$$

then $\mathcal{N}_{\mathfrak{p}}^{\mathfrak{q}}(h,t)\subset \mathcal{H}_{\lambda_0}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$.

Proof. Let $h\in \mathcal{N}_{\mathfrak{p}}^{\mathfrak{q}}(h,t)$, we have from Definition 1.3

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} \mathfrak{m}\,|a_{\mathfrak{m}}-c_{\mathfrak{m}}| \leqq \lambda,$$

which directly leads to the inequality of the given coefficients

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} |a_{\mathfrak{m}}-c_{\mathfrak{m}}| \leqq \frac{\lambda}{[\mathfrak{p}+1]_{\mathfrak{q}}}.$$

Since $t\in \mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$, according to inequality (8) we conclude that

$$\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty} c_{\mathfrak{m}} \leqq \frac{\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}.$$

Thus, by defining the family $\mathcal{H}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$, it becomes clear that

$$\left|\frac{h(\xi)}{t(\xi)}-1\right| < \frac{\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}|a_{\mathfrak{m}}-c_{\mathfrak{m}}|}{1-\sum_{\mathfrak{m}=\mathfrak{p}+1}^{\infty}c_{\mathfrak{m}}} \leqq \frac{\lambda}{[\mathfrak{p}+1]_{\mathfrak{q}}}\cdot\frac{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}}{[\mathfrak{p}+1]_{\mathfrak{q}}(\chi_1\gamma+1)\left(\frac{[\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}{[1+\mathfrak{p}+\varepsilon]_{\mathfrak{q}}}\right)^{s}-\gamma[\mathfrak{p}]_{\mathfrak{q}}(\chi_1+\chi_2)}$$
$$= 1-\lambda_0,$$

since λ_0 is defined precisely according to equation (21), we conclude by definition that $h(\xi)\in \mathcal{H}_{\lambda_0}(\chi_1,\chi_2,\gamma,s,\mathfrak{p},\mathfrak{q})$ when λ_0 as in (21), and thus the proof is complete. □

Conclusion

This paper addressed the development of a new family of $\mathfrak{q}$-starlike multivalent functions involving a new integral operator. Some of geometric properties, such as initial coefficients, distortion and growth theorems, extreme points, as well as λ-neighborhood, and radius theory, were investigated. These investigations lead to the exploration of a range of other geometric properties, such as Hankel determinants and partial sums. The Bernardi operator is a powerful tool in the field of differential subordination, used to analyze the behavior of functions in terms of their dependence on variables. This application also contributes to the development of new theories of stability and differential analysis, enhancing our understanding of complex mathematical and physical systems.

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