Results in Nonlinear Analysis 9 (2026) No. 1, 51–64
https://doi.org/10.31838/rna/2026.09.01.006
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Fekete–Szegö inequalities and initial coefficient bounds for Bi-univalent functions involving fractional q-differential operator subordination to q-hermite polynomials

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Abstract

In this paper, we introduce new subclasses of bi-univalent functions in the open unit disk defined via the q-fractional difference operator related to q-Hermite polynomials. Using subordination, we define the starlike class $\mathcal{B}_{\daleth}$ and the convex class $\mathcal{T}_{\daleth}$. For functions in these classes, we derive upper bounds for the first two Taylor coefficients $|a_2|$ and $|a_3|$, and we establish upper Fekete–Szegö inequalities for $\left|a_3 - \beta a_2^2\right|$ and $\left|a_3 - \mu a_2^2\right|$, extending classical results to the q-fractional setting.

We focus particularly on comparing the initial coefficients of these functions, providing insight into their geometric behavior and the influence of the q-fractional operator on the coefficient bounds.

Mathematics Subject Classification (2020): 28A80
Key words and Phrases: q-Hermite Polynomials; Fekete–Szegö; Starlike; Fractional q-calculus; Convex functions.

1. Introduction

Fractional q-calculus has recently emerged as a dynamic extension of classical calculus, capturing the interest of researchers in approximation theory and beyond. Its framework finds applications across a variety of fields, including optimal control, q-difference equations, hypergeometric series, quantum

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Received February 09, 2026; Accepted March 14, 2026; Online April 11, 2026

mechanics, and fractional subdiffusion processes [1–3]. Historically, Lupas [4] introduced p-calculus, while Jackson [5] established q-analogues of derivatives and integrals, laying the foundation for modern developments.

Let $\mathbb{A}$ be the class of analytic functions in the unit disk ∇ of the form

$$f(\zeta)=\zeta+\sum_{n=2}^{\infty}\mathsf{a}_n\zeta^n. \tag{1}$$

Within $\mathbb{A}$, the subclass $\mathcal{S}$ consists of normalized univalent functions satisfying $f(0)=0$ and $f'(0)=1$. Each $f\in\mathcal{S}$ has an inverse f^{-1} defined in $|w|<r_0(f)\geq 1/4$, with the expansion

$$g(w)=f^{-1}(w)=w-\mathsf{a}_2w^2+(2\mathsf{a}_2^2-\mathsf{a}_3)w^3-(5\mathsf{a}_2^3-5\mathsf{a}_2\mathsf{a}_3+\mathsf{a}_4)w^4+\cdots. \tag{2}$$

Bi-univalent functions are of particular interest in geometric function theory because both a function f and its inverse f^{-1} are univalent in the unit disk ∇. The set of all such functions is denoted by $\beth$.

Some classical examples of bi-univalent functions and their inverses are listed in Table 1.

Table 1: Examples of bi-univalent functions and their inverses.

Function	Expression	Inverse
f_1	$\dfrac{\zeta}{1-\zeta}$	$\dfrac{w}{1+w}$
f_2	$\dfrac{1}{2}\log\left(\dfrac{1+\zeta}{1-\zeta}\right)$	$\dfrac{e^{2w}-1}{e^{2w}+1}$
f_3	$-\log(1-\zeta)$	$\dfrac{e^{w}-1}{e^{w}}$

Recall that $f\in\mathbb{A}$ is given by (1). A fundamental operation in this setting is the convolution (or Hadamard product), which blends the coefficients of two analytic functions. For $f\ \ h\in\mathbb{A}$, with $h(\zeta)=\zeta+\sum_{n=2}^{\infty}b_n\zeta^n$, their convolution is defined as

$$(f\star h)(\zeta)=\zeta+\sum_{n=2}^{\infty}\mathsf{a}_nb_n\zeta^n.$$

This operation is extremely useful for combining functional properties in a coefficient-wise manner.

Another central concept is subordination, which provides a way to compare analytic functions. We say f is subordinate to g, written $f(\zeta)\prec g(\zeta)$, if there exists an analytic mapping $w:\nabla\to\nabla$ with $w(0)=0$ and $|w(\zeta)|<1$ such that $f(\zeta)=g(w(\zeta))$. If g is univalent, subordination reduces to the intuitive condition $f(0)=g(0)$ and $f(\nabla)\subset g(\nabla)$, providing a clear geometric interpretation of inclusion [6–8].

Lewin [9] initiated the study of the bi-univalent class $\beth$, showing that $|\mathsf{a}_2|<1.51$. Later works by Brannan and Clunie [10–12] and Netanyahu [13] extended these investigations, providing bounds for $|\mathsf{a}_2|$ and $|\mathsf{a}_3|$ in various subclasses. Despite these advances, estimating $|\mathsf{a}_n|$ for $n\geq 4$ remains an open problem. Over the decades, many subclasses have been analyzed, and sharp or non-sharp bounds for the first few coefficients have been established in multiple contexts [14, 15].

In geometric function theory, starlike and convex functions are often characterized using subordination. A function f is starlike of order θ if

$$\frac{zf'(\zeta)}{f(\zeta)}\prec\frac{1+(1-2\theta)\zeta}{1-\zeta},\quad 0\leq\theta<1,$$

and convex of order θ if

$$1+\frac{zf''(\zeta)}{f'(\zeta)}\prec\frac{1+(1-2\theta)\zeta}{1-\zeta}.$$

Ma and Minda [16] introduced a flexible framework using an analytic function Φ with

$$\Re(\Phi(\zeta))>0,\quad \Phi(0)=1,\quad \Phi'(0)>0,$$

mapping the unit disk ∇ into a starlike domain with respect to 1 and symmetric about the real axis. This yields the generalized classes:

$$\mathcal{S}^*(\Phi)=\left\{f\in\mathbb{A}:\frac{zf'(\zeta)}{f(\zeta)}\prec\Phi(\zeta)\right\},\quad \mathcal{C}(\Phi)=\left\{f\in\mathbb{A}:1+\frac{zf''(\zeta)}{f'(\zeta)}\prec\Phi(\zeta)\right\}.$$

Several notable choices of Φ and the corresponding subclasses are summarized in Table 2, highlighting the diversity and flexibility of the Ma-Minda framework.

Table 2: Representative subclasses generated via Ma-Minda functions Φ

$\Phi(\zeta)$	Subclass (Reference)
$\sqrt{1+\zeta}$	$\mathcal{S}^*_L$ (Sokoł, 2009 [17]; Raza & Malik, 2013 [18])
$\zeta+\sqrt{1+\zeta^2}$	$\mathcal{S}^*_l$ (Raina & Sokoł, 2015 [19])
$1+\zeta-\frac{1}{3}\zeta^2+\frac{1}{9}\zeta^3$	$\mathcal{S}^*_H$ (Tayyah and Atshan, 2025 [20])
$1+\sin(\zeta)$	$\mathcal{S}^*_{\sin}$ (Arif et al., 2019 [21])

Jackson's q-derivative gives a discrete version of the usual derivative. For $f\in\mathbb{A}$, it is defined by

$$\mathbb{D}_q f(\zeta)=\begin{cases}\dfrac{f(\zeta)-f(q\zeta)}{(1-q)\zeta}, & z\neq 0,\\ f'(0), & z=0,\end{cases}\quad q\in(0,1), \tag{3}$$

with $\lim\limits_{q\to 1^-}\mathbb{D}_q f(\zeta)=f'(\zeta)$. For powers,

$$\mathbb{D}_q\zeta^n=[n]_q\zeta^{n-1},\quad [n]_q=1+q+\cdots+q^{n-1}.$$

This definition follows Jackson's original work on q-calculus [22].

The q-derivative is linear and satisfies q-versions of the product and quotient rules. Higher-order derivatives follow the q-Leibniz formula:

$$\mathbb{D}_q^{(n)}(fg)(\zeta)=\sum_{k=0}^{n}\binom{n}{k}_q\mathbb{D}_q^{(k)}f(q^{n-k}\zeta)\mathbb{D}_q^{(n-k)}g(q^k\zeta).$$

Using $\mathbb{D}_q$, q-starlike and q-convex functions are defined as

$$S_q^*=\left\{f\in\mathbb{A}:\Re\left(\frac{z\mathbb{D}_q f(\zeta)}{f(\zeta)}\right)>0\right\},\quad K_q=\left\{f\in\mathbb{A}:\Re\left(\frac{\mathbb{D}_q(\zeta\mathbb{D}_q f(\zeta))}{\mathbb{D}_q f(\zeta)}\right)>0\right\}, \tag{4}$$

which reduce to the classical starlike and convex classes as $q\to 1^-$.

In terms of subordination, we have

$$S_{\mathsf{q}}^{*}(\phi)=\left\{f\in\mathbb{A}:\frac{z\mathbb{D}_{\mathsf{q}}f(\zeta)}{f(\zeta)}\prec\phi(\zeta)\right\},\quad K_{\mathsf{q}}(\phi)=\left\{f\in\mathbb{A}:\frac{\mathbb{D}_{\mathsf{q}}(\zeta\mathbb{D}_{\mathsf{q}}f(\zeta))}{\mathbb{D}_{\mathsf{q}}f(\zeta)}\prec\phi(\zeta)\right\},$$

providing a generalized framework for q-analogs of starlike and convex functions [23, 25].

Recently, there has been considerable interest in q-analogues of analytic bi-univalent function classes. For example, some authors [26] investigated a generalized q-differential operator using q -hypergeometric functions and derived various applications. Other works [2, 27] studied q-starlike functions associated with generalized conic domains. Extensions to multivalent functions via key operators have also been explored [23, 24, 28], along with q-starlike classes over conic regions and q-Noor integral operators [29]. Additionally, the theory of fractional q-calculus and its applications to geometric function classes in complex analysis has been examined extensively [27].

Let q be fixed with $0<\mathsf{q}<1$ and $n\in\mathbb{N}$. Some useful notations from [5] are

$$(a;\mathsf{q})_n=\prod_{i=0}^{n-1}(1-\mathsf{q}^i a),\quad (a;\mathsf{q})_0=1,\quad (a;\mathsf{q})_\infty=\prod_{i=0}^{\infty}(1-\mathsf{q}^i a).$$

The q-analogue of the Gamma function is defined by

$$\Gamma_{\mathsf{q}}(1-\ltimes)=\frac{(\mathsf{q};\mathsf{q})_\infty}{(\mathsf{q}^{1-\ltimes};\mathsf{q})_\infty}(1-\mathsf{q})^{\ltimes},\quad 0<\ltimes<1.$$

The q-analogue of the difference operator of non-integer order $\ltimes$ is given as

$$\mathbb{D}_{\mathsf{q}}^{\ltimes}f(\zeta)=\frac{1}{(1-\mathsf{q})^{\ltimes}\zeta^{\ltimes}}\sum_{n=0}^{\infty}\frac{(\mathsf{q}^{-\ltimes};\mathsf{q})_n}{(\mathsf{q};\mathsf{q})_n}\mathsf{q}^n f(\mathsf{q}^n\zeta),\quad \ltimes\neq 0,\mathsf{q}\in(0,1).$$

Notice that as $|q|<1$, $\ltimes\to 1$, we recover the usual q-derivative: $\mathbb{D}_{\mathsf{q}}^{\ltimes}f(\zeta)\to\mathbb{D}_{\mathsf{q}}f(\zeta)$, was introduced in [30].

For monomials ζ^n with $n\in\mathbb{N}$, we have

$$\mathbb{D}_{\mathsf{q}}^{\ltimes}\zeta^n=\frac{(\mathsf{q};\mathsf{q})_n}{(\mathsf{q}^{1-\ltimes};\mathsf{q})_n}\zeta^{n-\ltimes}\Gamma_{\mathsf{q}}(1-\ltimes),\quad 0<\ltimes<1. \tag{6}$$

For $f\in\mathbb{A}$, the q-fractional difference operator can be expressed as

$$\frac{\zeta^{\ltimes}\Gamma_{\mathsf{q}}(1-\ltimes)[1-\mathsf{q}^{1-\ltimes}]}{1-q}\mathbb{D}_{\mathsf{q}}^{\ltimes}f(\zeta)=f(\zeta)\star I_{\mathsf{q}}^{\ltimes}(\zeta)=I_{\mathsf{q}}^{\ltimes}(\zeta),\quad 0<\ltimes<1,$$

where

$$I_{\mathsf{q}}^{\ltimes}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{(\mathsf{q};\mathsf{q})_n(1-\mathsf{q}^{1-\ltimes})}{(\mathsf{q}^{1-\ltimes};\mathsf{q})_n(1-\mathsf{q})}\zeta^n,\quad 0<\ltimes<1,$$

defines a linear operator

$$I_{\mathsf{q}}^{\ltimes}:\mathbb{A}\to\mathbb{A},\quad f\mapsto\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta)=f(\zeta)\star I_{\mathsf{q}}^{\ltimes}(\zeta).$$

In terms of coefficients, this convolution gives

$$\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta)=f(\zeta)\star I_{\mathsf{q}}^{\ltimes}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{(\mathsf{q};\mathsf{q})_n(1-\mathsf{q}^{1-\ltimes})}{(\mathsf{q}^{1-\ltimes};\mathsf{q})_n(1-\mathsf{q})}\mathsf{a}_n\zeta^n=\zeta+\sum_{n=2}^{\infty}\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)\mathsf{a}_n\zeta^n, \tag{7}$$

where

$$\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)=\frac{(\mathsf{q};\mathsf{q})_n(1-\mathsf{q}^{1-\ltimes})}{(\mathsf{q}^{1-\ltimes};\mathsf{q})_n(1-\mathsf{q})},\quad 0<\ltimes<1.$$

This operator preserves analyticity and the normalization of functions in $\mathbb{A}$, acting via convolution (Hadamard product) as a q-fractional analogue of classical integral operators [26].

The q-Hermite polynomials, first introduced by Rogers [31] (see also [32–37]), are an important family in the study of q-orthogonal polynomials and special functions. They can be defined through the generating function

$$\mathcal{L}_k(s|q) = \sum_{k=0}^{\infty} H_k(x;q)\frac{t^k}{(q;q)_k} = \prod_{k=0}^{\infty}\frac{1}{1-2xtq^k+t^2q^{2k}}, \qquad 0<q<1. \tag{8}$$

Their q-derivative satisfies

$$\mathbb{D}_q\{\mathcal{L}_{k+1}(s\,|\,q)\} = [k]_q\mathcal{L}_k(s\,|\,q), \tag{9}$$

and they follow the recurrence

$$t\mathcal{L}_k(s\,|\,q) = \mathcal{L}_{k+1}(s\,|\,q) + [k]_q\mathcal{L}_{k-1}(s\,|\,q), \tag{10}$$

with initial conditions

$$\mathcal{L}_0(s\,|\,q) = 1, \qquad \mathcal{L}_{-1}(s\,|\,q) = 0.$$

From this, the first few polynomials are

$$\begin{aligned}
&\mathcal{L}_1(s\,|\,q) = s,\\
&\mathcal{L}_2(s\,|\,q) = s^2-1,\\
&\mathcal{L}_3(s\,|\,q) = s^3-(2+q)s,\\
&\mathcal{L}_4(s\,|\,q) = s^4-(3+2q+q^2)s^2+(1+q+q^2).
\end{aligned}$$

These formulas show the neat recursive pattern of the q-Hermite polynomials, which smoothly turn into the classical Hermite polynomials when $q \to 1^-$.

Next [38], we introduce the q-Babalola convolution operator, which will be used in the upcoming definitions. Let

$$\mathcal{H}(\zeta, s, q) = \sum_{k=0}^{\infty}\mathcal{L}_k(s\,|\,q)\zeta^k.$$

Remark 1.1. The q-Hermite polynomials connect several classical families: for q = 1, they reduce to the standard Hermite polynomials, $\mathcal{L}_k(s\,|\,1) = \mathcal{L}_k^{c}(s)$; for q = 0, they become the Chebyshev polynomials of the first kind, $U_k(s/2)$, which satisfy

$$2sU_k(s) = U_{k-1}(s) + U_{k+1}(s), \qquad U_0(s) = 1, \quad U_{-1}(s) = 0.$$

Further details are given in [38].

2. Coefficient Estimates and Fekete-Szegö Inequalities for the Class $\mathcal{B}_{\beth}(s,q,\ltimes)$

In geometric function theory, estimating the initial coefficients of analytic or bi-univalent functions is important for understanding their geometric behavior and applications such as growth and distortion bounds. For the starlike class $\mathcal{B}_{\beth}(s,q,\ltimes)$, we derive bounds for the first two Taylor coefficients $|a_2|$ and $|a_3|$ and establish Fekete–Szegö inequalities for $\left|a_3-\beta a_2^2\right|$, extending classical results to the q-fractional setting.

A function $f \in \beth$ given by (1) belongs to the starlike class $\mathcal{B}_{\beth}(s,q,\ltimes)$ if

$$\frac{z(\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta))'}{\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta)} \prec \mathcal{H}(\zeta,s,\mathsf{q}), \tag{11}$$

and

$$\frac{w(\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w))'}{\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w)} \prec \mathcal{H}(w,s,\mathsf{q}), \tag{12}$$

where g is the inverse function of f.

Corollary 2.1. *If, instead,* $\mathsf{q}\to 1^-$ *(with* $\ltimes$ *fixed or* $\ltimes\to 1$ *as well), then the coefficient multipliers satisfy* $\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)\to 1$ *(in the appropriate limit) and the operator* $\mathcal{J}_{\ltimes}^{\mathsf{q}}$ *converges to the identity or to the classical differential operator according to the mode of passage to the limit. Consequently, for* $f\in\mathcal{B}_{\beth}(s,\mathsf{q},\ltimes)$ *one obtains in the limit*

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec \mathcal{H}(\zeta,s,1), \qquad \frac{wg'(w)}{g(w)} \prec \mathcal{H}(w,s,1),$$

when the limits produce the ordinary derivative (e.g. $\ltimes\to 1$ *followed by* $\mathsf{q}\to 1^-$*).*

Theorem 2.2. *A function* $f\in\beth$ *given by (1) belongs to the class* $\mathcal{B}_{\beth}(s,\mathsf{q},\ltimes)$ *if it satisfies the following conditions. Assume that* $0<\mathsf{q}<1$*, the order* $\ltimes$ *is non-integer, and* $s\in(0.5,1)$*. Then the following coefficient bounds hold:*

$$|\mathsf{a}_2| \le \sqrt{\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{\left|2\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)-[\mathbb{B}_{\ltimes}^{\mathsf{q}}(2)]^2-\left(\mathcal{L}_2(s\,|\,\mathsf{q})-\mathcal{L}_1(s\,|\,\mathsf{q})\right)\left(\dfrac{\mathbb{B}_{\ltimes}^{\mathsf{q}}(2)}{\mathcal{L}_1(s\,|\,\mathsf{q})}\right)^2\right|}}, \tag{13}$$

and

$$|\mathsf{a}_3| \le \frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}+\left(\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{\mathbb{B}_{\ltimes}^{\mathsf{q}}(2)}\right)^2. \tag{14}$$

Proof. Let $f\in\beth$ be defined by (1) and assume f is in the class $\mathcal{B}_{\beth}(s,\mathsf{q},\ltimes)$. Consider the functional relations

$$\frac{z(\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta))'}{\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta)} = \mathcal{N}(\Gamma(\zeta),s,\mathsf{q}), \qquad \frac{w(\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w))'}{\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w)} = \mathcal{N}(\Lambda(w),s,\mathsf{q}), \tag{15}$$

where g is the inverse of f. Let $p(\zeta),c(w)\in\mathcal{P}$ be analytic functions:

$$p(\zeta)=\frac{1+\Gamma(\zeta)}{1-\Gamma(\zeta)}=1+p_1\zeta+p_2\zeta^2+p_3\zeta^3+\ldots, \qquad \Gamma(\zeta)=\frac{p(\zeta)-1}{p(\zeta)+1}, \tag{16}$$

$$c(w)=\frac{1+\Lambda(w)}{1-\Lambda(w)}=1+\mathsf{c}_1w+\mathsf{c}_2w^2+\mathsf{c}_3w^3+\ldots, \qquad \Lambda(w)=\frac{c(w)-1}{c(w)+1}. \tag{17}$$

Expanding in series gives

$$\Gamma(\zeta)=\frac{1}{2}\Big[p_1\zeta+\Big(p_2-\frac{p_1^2}{2}\Big)\zeta^2+\Big(p_3-p_1p_2+\frac{p_1^3}{4}\Big)\zeta^3+\ldots\Big], \tag{18}$$

$$\Lambda(w)=\frac{1}{2}\Big[\mathsf{c}_1w+\Big(\mathsf{c}_2-\frac{\mathsf{c}_1^2}{2}\Big)w^2+\Big(\mathsf{c}_3-\mathsf{c}_1\mathsf{c}_2+\frac{\mathsf{c}_1^3}{4}\Big)w^3+\ldots\Big]. \tag{19}$$

Using the generating function $\mathcal{N}(\zeta, s, \mathrm{q})$ from (8), we get

$$\mathcal{N}(\Gamma(\zeta), s, \mathrm{q}) = 1 + \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2} p_1 \zeta + \Big[\frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\Big(p_2 - \frac{p_1^2}{2}\Big) + \frac{\mathcal{L}_2(s\,|\,\mathrm{q})}{4} p_1^2\Big]\zeta^2 + \ldots, \tag{20}$$

$$\mathcal{N}(\Lambda(w), s, \mathrm{q}) = 1 + \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2} \mathrm{c}_1 w + \Big[\frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\Big(\mathrm{c}_2 - \frac{\mathrm{c}_1^2}{2}\Big) + \frac{\mathcal{L}_2(s\,|\,\mathrm{q})}{4} \mathrm{c}_1^2\Big] w^2 + \ldots. \tag{21}$$

On the other hand, applying the operator $\mathcal{J}_{\ltimes}^{\mathrm{q}}$ to *f* and *g* gives

$$\frac{z(\mathcal{J}_{\ltimes}^{\mathrm{q}} f(\zeta))'}{\mathcal{J}_{\ltimes}^{\mathrm{q}} f(\zeta)} = 1 + \mathbb{B}_{\ltimes}^{\mathrm{q}}(2)\mathrm{a}_2\zeta + \Big(-[\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)]^2\mathrm{a}_2^2 + 2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3)\mathrm{a}_3\Big)\zeta^2 + \ldots, \tag{22}$$

$$\frac{w(\mathcal{J}_{\ltimes}^{\mathrm{q}} g(w))'}{\mathcal{J}_{\ltimes}^{\mathrm{q}} g(w)} = 1 - \mathbb{B}_{\ltimes}^{\mathrm{q}}(2)\mathrm{a}_2 w + \Big([4\mathbb{B}_{\ltimes}^{\mathrm{q}}(3) - (\mathbb{B}_{\ltimes}^{\mathrm{q}}(2))^2]\mathrm{a}_2^2 - 2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3)\mathrm{a}_3\Big)w^2 + \ldots. \tag{23}$$

Comparing the coefficients in these expansions, we obtain

$$\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)\mathrm{a}_2 = \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2} p_1, \tag{24}$$

$$2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3)\mathrm{a}_3 - [\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)]^2\mathrm{a}_2^2 = \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\Big(p_2 - \frac{p_1^2}{2}\Big) + \frac{\mathcal{L}_2(s\,|\,\mathrm{q})}{4} p_1^2, \tag{25}$$

$$-\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)\mathrm{a}_2 = \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\mathrm{c}_1, \tag{26}$$

$$[4\mathbb{B}_{\ltimes}^{\mathrm{q}}(3) - (\mathbb{B}_{\ltimes}^{\mathrm{q}}(2))^2]\mathrm{a}_2^2 - 2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3)\mathrm{a}_3 = \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\Big(\mathrm{c}_2 - \frac{\mathrm{c}_1^2}{2}\Big) + \frac{\mathcal{L}_2(s\,|\,\mathrm{q})}{4}\mathrm{c}_1^2. \tag{27}$$

By referring to equations (24) and (26), we obtain

$$p_1 = -\mathrm{c}_1, \tag{28}$$

and

$$2\big[\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)\big]^2\mathrm{a}_2^2 = \left(\frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}\right)^2\big(p_1^2 + \mathrm{c}_1^2\big). \tag{29}$$

By adding equations (25) and (27), we obtain

$$2\big(2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3) - [\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)]^2\big)\mathrm{a}_2^2 = \left(\frac{\mathcal{L}_2(s\,|\,\mathrm{q})}{4} - \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{4}\right)(p_1^2 + \mathrm{c}_1^2) + \frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{2}(p_2 + \mathrm{c}_2). \tag{30}$$

Using equation (29), we obtain

$$\mathrm{a}_2^2 = \frac{\mathcal{L}_1(s\,|\,\mathrm{q})(p_2 + \mathrm{c}_2)}{4\left[2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3) - (\mathbb{B}_{\ltimes}^{\mathrm{q}}(2))^2 - (\mathcal{L}_2(s\,|\,\mathrm{q}) - \mathcal{L}_1(s\,|\,\mathrm{q}))\left(\dfrac{\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)}{\mathcal{L}_1(s\,|\,\mathrm{q})}\right)^2\right]}. \tag{31}$$

Now, we determine an upper bound for $|a_2|$. By applying Lemma 1, we obtain

$$|\mathrm{a}_2| \le \sqrt{\frac{\mathcal{L}_1(s\,|\,\mathrm{q})}{\left|2\mathbb{B}_{\ltimes}^{\mathrm{q}}(3) - [\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)]^2 - \big(\mathcal{L}_2(s\,|\,\mathrm{q}) - \mathcal{L}_1(s\,|\,\mathrm{q})\big)\left(\dfrac{\mathbb{B}_{\ltimes}^{\mathrm{q}}(2)}{\mathcal{L}_1(s\,|\,\mathrm{q})}\right)^2\right|}}.$$

By subtracting equations (25) and (27), we obtain

$$4\mathbb{B}^{q}_{\ltimes}(3)\left(a_2 - a_2^2\right) = \frac{\mathcal{L}_1(s\,|\,q)}{2}(p_2 - c_2). \tag{32}$$

Performing some straightforward algebraic manipulations, we get

$$a_2 = \frac{\mathcal{L}_1(s\,|\,q)}{8\mathbb{B}^{q}_{\ltimes}(3)}(p_2 - c_2) + a_2^2. \tag{33}$$

Substituting the value of a_2^2 from equation (29), we obtain

$$a_2 = \frac{\mathcal{L}_1(s\,|\,q)}{8\mathbb{B}^{q}_{\ltimes}(3)}(p_2 - c_2) + \left(\frac{\mathcal{L}_1(s\,|\,q)}{2}\right)^2 \frac{p_1^2 + c_1^2}{2\left[\mathbb{B}^{q}_{\ltimes}(2)\right]^2}. \tag{34}$$

Finally, by applying Lemma 1 together with the triangle inequality, we arrive at the following upper bound:

$$|a_2| \le \frac{\mathcal{L}_1(s\,|\,q)}{2\mathbb{B}^{q}_{\ltimes}(3)} + \left(\frac{\mathcal{L}_1(s\,|\,q)}{\mathbb{B}^{q}_{\ltimes}(2)}\right)^2. \tag{35}$$

□

Theorem 2.3. *Let $f \in \gimel$ be given by (1) and belong to the class $\mathcal{B}_{\gimel}(s, q, \ltimes)$. Assume that $0 < q < 1$, the order $\ltimes$ is non-integer, and $s \in (0.5, 1)$. Then the coefficients satisfy the following bound:*

$$|a_3 - \beta a_2^2| \le \begin{cases} \mathcal{L}_1(s\,|\,q)\left|\dfrac{1}{4\mathbb{B}^{q}_{\ltimes}(3)}\right|, & \text{if } |\ell(\beta)| \le \left|\dfrac{1}{8\mathbb{B}^{q}_{\ltimes}(3)}\right|, \\ [5pt]2\mathcal{L}_1(s\,|\,q)\,|\ell(\beta)|, & \text{if } |\ell(\beta)| \ge \left|\dfrac{1}{8\mathbb{B}^{q}_{\ltimes}(3)}\right|, \end{cases} \tag{36}$$

where

$$\ell(\beta) = \frac{1-\beta}{4\left[2\mathbb{B}^{q}_{\ltimes}(3) - \left(\mathbb{B}^{q}_{\ltimes}(2)\right)^2 - \left(\mathcal{L}_2(s\,|\,q) - \mathcal{L}_1(s\,|\,q)\right)\left(\dfrac{\mathbb{B}^{q}_{\ltimes}(2)}{\mathcal{L}_1(s\,|\,q)}\right)^2\right]}.$$

Proof. By referring to equation (33), we have

$$a_2 - \beta a_2^2 = \frac{\mathcal{L}_1(s\,|\,q)}{8\mathbb{B}^{q}_{\ltimes}(3)}(p_2 - c_2) + (1-\beta)a_2^2. \tag{37}$$

Substituting the value of a_2^2 from the corresponding equation (31), we get

$$a_3 - \beta a_2^2 = \frac{\mathcal{L}_1(s\,|\,q)(p_2 - c_2)}{8\mathbb{B}^{q}_{\ltimes}(3)} + \frac{\mathcal{L}_1(s\,|\,q)(1-\beta)(p_2 + c_2)}{4\left[2\mathbb{B}^{q}_{\ltimes}(3) - (\mathbb{B}^{q}_{\ltimes}(2))^2 - (\mathcal{L}_2(s\,|\,q) - \mathcal{L}_1(s\,|\,q))\left(\dfrac{\mathbb{B}^{q}_{\ltimes}(2)}{\mathcal{L}_1(s\,|\,q)}\right)^2\right]}. \tag{38}$$

After some simplifications and algebraic manipulations, we arrive at

$$a_2 - \beta a_2^2 = \mathcal{L}_1(s\,|\,q)\left[\left(\ell(\beta) + \frac{1}{8\mathbb{B}^{q}_{\ltimes}(3)}\right)p_2 + \left(\ell(\beta) - \frac{1}{8\mathbb{B}^{q}_{\ltimes}(3)}\right)c_2\right],$$

where

$$\ell(\beta)=\frac{1-\beta}{4\left[2\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)-\left(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2)\right)^{2}-\left(\mathcal{L}_{2}(s\,|\,\mathsf{q})-\mathcal{L}_{1}(s\,|\,\mathsf{q})\right)\left(\dfrac{\mathbb{B}_{\ltimes}^{\mathsf{q}}(2)}{\mathcal{L}_{1}(s\,|\,\mathsf{q})}\right)^{2}\right]}.$$

□

Corollary 2.4. *Under the hypotheses of theorem 2.2, if* $\mathsf{q}\to 1^{-}$ *(with* $\ltimes$ *fixed or with* $\ltimes\to 1$ *as well) so that* $\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)\to 1$ *for every n, then the operator* $\mathcal{J}_{\ltimes}^{\mathsf{q}}$ *converges to the identity or to the classical differential operator according to the mode of passage to the limit. Consequently the coefficient estimates (13–36) reduce to their classical forms obtained by setting* $\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)=1$ *and replacing* $\mathcal{L}_{i}(s\,|\,\mathsf{q})$ *by their limits* $\mathcal{L}_{i}(s)$ *(when these limits exist). In particular:*

$$|\mathsf{a}_{2}|\leq\sqrt{\frac{\mathcal{L}_{1}(s)}{\left|1-\dfrac{\mathcal{L}_{2}(s)-\mathcal{L}_{1}(s)}{(\mathcal{L}_{1}(s))^{2}}\right|}},$$

$$|\mathsf{a}_{3}|\leq\frac{\mathcal{L}_{1}(s)}{2}+\left(\mathcal{L}_{1}(s)\right)^{2},$$

and for any real or complex parameter β *the Fekete-Szegö inequality* (36) *becomes*

$$|\mathsf{a}_{3}-\beta\mathsf{a}_{2}^{2}|\leq\begin{cases}\dfrac{\mathcal{L}_{1}(s)}{4}, & \text{if } |\ell(s,\beta)|\leq\dfrac{1}{8},\\ [6pt]2\mathcal{L}_{1}(s)|\,\ell(s,\beta)\,|, & \text{if } |\ell(s,\beta)|\geq\dfrac{1}{8},\end{cases}$$

where

$$\ell(s,\beta)=\frac{1-\beta}{4\left[1-\dfrac{\mathcal{L}_{2}(s)-\mathcal{L}_{1}(s)}{(\mathcal{L}_{1}(s))^{2}}\right]}=\frac{(1-\beta)(\mathcal{L}_{1}(s))^{2}}{4\left[(\mathcal{L}_{1}(s))^{2}-\left(\mathcal{L}_{2}(s)-\mathcal{L}_{1}(s)\right)\right]}.$$

In words: all occurrences of $\mathbb{B}_{\ltimes}^{\mathsf{q}}(n)$ *in* (13) *and* (36) *are replaced by* 1, *and* $\mathcal{L}_{i}(s\,|\,\mathsf{q})$ *are replaced by their classical limits* $\mathcal{L}_{i}(s)$, *yielding the corresponding classical coefficient bounds.*

3. Coefficient Estimates and Fekete-Szegö Inequalities for the Class $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$

In geometric function theory, studying the initial coefficients of analytic or bi-univalent functions in convex classes provides insight into their growth, distortion, and covering properties. For the convex class $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$, we focus on deriving bounds for the first two Taylor coefficients $|a_{2}|$ and $|a_{3}|$, and we establish Fekete–Szegö inequalities for $\left|a_{3}-\mu a_{2}^{2}\right|$, extending classical results to the q-fractional setting.

Definition 3.1. A function $f\in\beth$ given by (1) belongs to the convex class $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$ if

$$1+\frac{z(\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta))''}{(\mathcal{J}_{\ltimes}^{\mathsf{q}}f(\zeta))'}\prec\mathcal{H}(\zeta,s,\mathsf{q}),\tag{39}$$

and

$$1+\frac{w(\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w))''}{(\mathcal{J}_{\ltimes}^{\mathsf{q}}g(w))'}\prec\mathcal{H}(w,s,\mathsf{q}),\tag{40}$$

where g is the inverse function of f.

Corollary 3.2. *If the limits* $|\mathsf{q}|<1$, $\ltimes\to 1$ *and then* $\mathsf{q}\to 1^{-}$ *(or vice versa) are taken, the class* $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$ *reduces to the classical convex class. In this case, for* $f\in\beth$ *one has*

$$1+\frac{\zeta f''(\zeta)}{f'(\zeta)}\prec\mathcal{H}(\zeta,s,1),\qquad 1+\frac{wg''(w)}{g'(w)}\prec\mathcal{H}(w,s,1),$$

where g is the inverse of f, and all coefficient or Fekete–Szegö type estimates for $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$ *reduce to their classical forms by setting* $\mathbb{B}^{\mathsf{q}}_{\ltimes}(n)=1$ *and replacing* $\mathcal{L}_i(s\,|\,\mathsf{q})$ *with* $\mathcal{L}_i(s)$.

Theorem 3.3. *A function* $f\in\beth$ *given by* (1) *belongs to the class* $\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$ *if it satisfies the following conditions. Assume that* $0<\mathsf{q}<1$, *the order* $\ltimes$ *is non-integer, and* $s\in(0.5,1)$. *Then the following coefficient bounds hold:*

$$|\mathsf{a}_2|\le\sqrt{\frac{2\left(\mathcal{L}_1(s\,|\,\mathsf{q})\right)^3}{\left(\mathcal{L}_1(s\,|\,\mathsf{q})\right)^2\left[12\mathbb{B}^{\mathsf{q}}_{\ltimes}(3)-8\left(\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)\right)^2\right]-8\left(\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)\right)^2\left(\mathcal{L}_2(s\,|\,\mathsf{q})-\mathcal{L}_1(s\,|\,\mathsf{q})\right)}},\tag{41}$$

and

$$|\mathsf{a}_3|\le\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{6\mathbb{B}^{\mathsf{q}}_{\ltimes}(3)}+\frac{(\mathcal{L}_1(s\,|\,\mathsf{q}))^2}{4[\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)]^2}.\tag{42}$$

Proof. Let $f\in\beth$ be defined by (1) with $f\in\mathcal{T}_{\beth}(s,\mathsf{q},\ltimes)$. Following section 2, we consider

$$1+\frac{z(\mathcal{J}^{\mathsf{q}}_{\ltimes}f(\zeta))''}{(\mathcal{J}^{\mathsf{q}}_{\ltimes}f(\zeta))'}=\mathcal{N}(\varGamma(\zeta),s,\mathsf{q}),\qquad 1+\frac{w(\mathcal{J}^{\mathsf{q}}_{\ltimes}g(w))''}{(\mathcal{J}^{\mathsf{q}}_{\ltimes}g(w))'}=\mathcal{N}(\varLambda(w),s,\mathsf{q}),\tag{43}$$

where $g=f^{-1}$ and $\varGamma(\zeta),\varLambda(w)\in\nabla$ are analytic functions as defined in section 2.

Using the generating function $\mathcal{N}(\zeta,s,\mathsf{q})$ from (8), we obtain the desired relations for coefficient estimates.

$$\mathcal{N}(\varGamma(\zeta),s,\mathsf{q})=1+\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2}p_1z+\left[\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{\mathcal{L}_2(s\,|\,\mathsf{q})}{4}p_1^2\right]\zeta^2+\cdots,\tag{44}$$

$$\mathcal{N}(\varLambda(w),s,\mathsf{q})=1+\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2}\mathsf{c}_1w+\left[\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2}\left(\mathsf{c}_2-\frac{\mathsf{c}_1^2}{2}\right)+\frac{\mathcal{L}_2(s\,|\,\mathsf{q})}{4}\mathsf{c}_1^2\right]w^2+\cdots.\tag{45}$$

On the other hand, applying the q-operator $\mathcal{J}^{\mathsf{q}}_{\ltimes}$ to f and g yields the following series expansions:

$$1+\frac{z(\mathcal{J}^{\mathsf{q}}_{\ltimes}f(\zeta))''}{(\mathcal{J}^{\mathsf{q}}_{\ltimes}f(\zeta))'}=1+2\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)\mathsf{a}_2z+\left(-4[\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)]^2\mathsf{a}_2^2+6\mathbb{B}^{\mathsf{q}}_{\ltimes}(3)\mathsf{a}_3\right)\zeta^2+\cdots,\tag{46}$$

and

$$1+\frac{w(\mathcal{J}^{\mathsf{q}}_{\ltimes}g(w))''}{(\mathcal{J}^{\mathsf{q}}_{\ltimes}g(w))'}=1-2\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)\mathsf{a}_2w+\left((12\mathbb{B}^{\mathsf{q}}_{\ltimes}(3)-4[\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)]^2)\mathsf{a}_2^2-6\mathbb{B}^{\mathsf{q}}_{\ltimes}(3)\mathsf{a}_3\right)w^2+\cdots.\tag{47}$$

By comparing the coefficients of ζ in equations (48) and (50), and the coefficients of w in equations (49) and (51), we obtain

$$2\mathbb{B}^{\mathsf{q}}_{\ltimes}(2)\mathsf{a}_2=\frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{2}p_1,\tag{48}$$

$$6\mathbb{B}_{\varkappa}^{q}(3)a_3-4[\mathbb{B}_{\varkappa}^{q}(2)]^2a_2^2=\frac{\mathcal{L}_1(s\mid q)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{\mathcal{L}_2(s\mid q)}{4}p_1^2, \tag{49}$$

$$-2\mathbb{B}_{\varkappa}^{q}(2)a_2=\frac{\mathcal{L}_1(s\mid q)}{2}c_1, \tag{50}$$

and

$$(12\mathbb{B}_{\varkappa}^{q}(3)-4[\mathbb{B}_{\varkappa}^{q}(2)]^2)a_2^2-6\mathbb{B}_{\varkappa}^{q}(3)a_3=\frac{\mathcal{L}_1(s\mid q)}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{\mathcal{L}_2(s\mid q)}{4}c_1^2. \tag{51}$$

By referring to equations (48) and (50), we obtain

$$p_1=-c_1, \tag{52}$$

and

$$8\left[\mathbb{B}_{\varkappa}^{q}(2)\right]^2a_2^2=\left(\frac{\mathcal{L}_1(s\mid q)}{2}\right)^2\left(p_1^2+c_1^2\right)\Rightarrow a_2^2=\frac{[\mathcal{L}_1(s\mid q)]^2}{32[\mathbb{B}_{\varkappa}^{q}(2)]^2}(p_1^2+c_1^2). \tag{53}$$

By adding equations (49) and (51), we obtain

$$(12\mathbb{B}_{\varkappa}^{q}(3)-8[\mathbb{B}_{\varkappa}^{q}(2)]^2)a_2^2=\left(\frac{\mathcal{L}_2(s\mid q)}{4}-\frac{\mathcal{L}_1(s\mid q)}{4}\right)(p_1^2+c_1^2)+\frac{\mathcal{L}_1(s\mid q)}{2}(p_2+c_2). \tag{54}$$

Using equation (53), we obtain

$$a_2^2=\frac{\left(\frac{\mathcal{L}_1(s\mid q)}{2}\right)^3(p_2+c_2)}{\left(\frac{\mathcal{L}_1(s\mid q)}{2}\right)^2\left[12\mathbb{B}_{\varkappa}^{q}(3)-8\left(\mathbb{B}_{\varkappa}^{q}(2)\right)^2\right]-8\left(\mathbb{B}_{\varkappa}^{q}(2)\right)^2\left(\frac{\mathcal{L}_2(s\mid q)}{4}-\frac{\mathcal{L}_1(s\mid q)}{4}\right)}. \tag{55}$$

Now, we determine an upper bound for $|a_2|$. By applying Lemma 1, we obtain

$$|a_2|\leq\sqrt{\frac{2\left(\mathcal{L}_1(s\mid q)\right)^3}{\left(\mathcal{L}_1(s\mid q)\right)^2\left[12\mathbb{B}_{\varkappa}^{q}(3)-8\left(\mathbb{B}_{\varkappa}^{q}(2)\right)^2\right]-8\left(\mathbb{B}_{\varkappa}^{q}(2)\right)^2\left(\mathcal{L}_2(s\mid q)-\mathcal{L}_1(s\mid q)\right)}}.$$

By subtracting equations (49) and (51), we obtain

$$12\mathbb{B}_{\varkappa}^{q}(3)(a_3-a_2^2)=\frac{\mathcal{L}_1(s\mid q)}{2}(p_2-c_2). \tag{56}$$

Performing some straightforward algebraic manipulations, we get

$$a_2=\frac{\mathcal{L}_1(s\mid q)}{24\mathbb{B}_{\varkappa}^{q}(3)}(p_2-c_2)+a_2^2. \tag{57}$$

Substituting the value of a_2^2 from equation (53), we obtain

$$a_2=\frac{\mathcal{L}_1(s\mid q)}{24\mathbb{B}_{\varkappa}^{q}(3)}(p_2-c_2)+\frac{(\mathcal{L}_1(s\mid q))^2}{32[\mathbb{B}_{\varkappa}^{q}(2)]^2}(p_1^2+c_1^2). \tag{58}$$

Finally, by applying Lemma 1 together with the triangle inequality, we arrive at the following upper bound:

$$|a_3|\leq\frac{4\mathcal{L}_1(s\mid q)}{24\mathbb{B}_{\varkappa}^{q}(3)}+\frac{8(\mathcal{L}_1(s\mid q))^2}{32[\mathbb{B}_{\varkappa}^{q}(2)]^2}=\frac{\mathcal{L}_1(s\mid q)}{6\mathbb{B}_{\varkappa}^{q}(3)}+\frac{(\mathcal{L}_1(s\mid q))^2}{4[\mathbb{B}_{\varkappa}^{q}(2)]^2}.$$

□

Theorem 3.4. *Let* $f \in \beth$ *be given by* (1) *and belong to the class* $\mathcal{T}_{\beth}(s, \mathsf{q}, \ltimes)$. *Assume that* $0 < \mathsf{q} < 1$*, the order* $\ltimes$ *is non-integer, and* $s \in (0.5, 1)$. *Then the coefficients satisfy the following bound:*

$$|\mathsf{a}_3 - \mu \mathsf{a}_2^2| \leq \begin{cases} \mathcal{L}_1(s\,|\,\mathsf{q}) \left|\dfrac{1}{12\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}\right|, & \text{if } |\ell(\mu)| \leq \left|\dfrac{1}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}\right|, \\ 2\mathcal{L}_1(s\,|\,\mathsf{q})\,|\ell(\mu)|, & \text{if } |\ell(\mu)| \geq \left|\dfrac{1}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}\right|, \end{cases} \tag{59}$$

where

$$\ell(\mu) = \frac{(\mathcal{L}_1(s\,|\,\mathsf{q}))^2(1-\mu)}{8\left[(\mathcal{L}_1(s\,|\,\mathsf{q}))^2\left(3\mathbb{B}_{\ltimes}^{\mathsf{q}}(3) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2\right) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2(\mathcal{L}_2(s\,|\,\mathsf{q}) - \mathcal{L}_1(s\,|\,\mathsf{q}))\right]}.$$

Proof. By referring to equation (57), we have

$$\mathsf{a}_2 - \mu \mathsf{a}_2^2 = \frac{\mathcal{L}_1(s\,|\,\mathsf{q})}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}(p_2 - \mathsf{c}_2) + (1-\mu)\mathsf{a}_2^2. \tag{60}$$

Substituting the value of a_2^2 from the corresponding equation (55), we get

$$\mathsf{a}_3 - \mu \mathsf{a}_2^2 = \frac{\mathcal{L}_1(s\,|\,\mathsf{q})(p_2 - \mathsf{c}_2)}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)} + \frac{(\mathcal{L}_1(s\,|\,\mathsf{q}))^3(1-\mu)(p_2 + \mathsf{c}_2)}{8\left[(\mathcal{L}_1(s\,|\,\mathsf{q}))^2\left(3\mathbb{B}_{\ltimes}^{\mathsf{q}}(3) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2\right) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2(\mathcal{L}_2(s\,|\,\mathsf{q}) - \mathcal{L}_1(s\,|\,\mathsf{q}))\right]}. \tag{61}$$

After some simplifications and algebraic manipulations, we arrive at

$$\mathsf{a}_2 - \mu \mathsf{a}_2^2 = \mathcal{L}_1(s\,|\,\mathsf{q})\left[\left(\ell(\mu) + \frac{1}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}\right)p_2 + \left(\ell(\mu) - \frac{1}{24\mathbb{B}_{\ltimes}^{\mathsf{q}}(3)}\right)\mathsf{c}_2\right],$$

where

$$\ell(\mu) = \frac{(\mathcal{L}_1(s\,|\,\mathsf{q}))^2(1-\mu)}{8\left[(\mathcal{L}_1(s\,|\,\mathsf{q}))^2\left(3\mathbb{B}_{\ltimes}^{\mathsf{q}}(3) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2\right) - 2(\mathbb{B}_{\ltimes}^{\mathsf{q}}(2))^2(\mathcal{L}_2(s\,|\,\mathsf{q}) - \mathcal{L}_1(s\,|\,\mathsf{q}))\right]}.$$

□

Corollary 3.5. *Under the limits* $|\mathsf{q}| < 1$, $\ltimes \to 1$ *followed by* $\mathsf{q} \to 1^-$ *(or vice versa), the class* $\mathcal{T}_{\beth}(s, \mathsf{q}, \ltimes)$ *reduces to the classical convex class. In this case, for* $f \in \beth$ *the coefficient bounds* (41–42) *simplify by setting* $\mathbb{B}_{\ltimes}^{\mathsf{q}}(n) = 1$ *and replacing* $\mathcal{L}_i(s\,|\,\mathsf{q})$ *with* $\mathcal{L}_i(s)$*:*

$$|\mathsf{a}_2| \leq \sqrt{\frac{2(\mathcal{L}_1(s))^3}{4(\mathcal{L}_1(s))^2 - 8(\mathcal{L}_2(s) - \mathcal{L}_1(s))}} = \sqrt{\frac{(\mathcal{L}_1(s))^3}{2(\mathcal{L}_1(s))^2 - 4(\mathcal{L}_2(s) - \mathcal{L}_1(s))}},$$

$$|\mathsf{a}_3| \leq \frac{\mathcal{L}_1(s)}{6} + \frac{(\mathcal{L}_1(s))^2}{4},$$

$$|\mathsf{a}_3 - \mu \mathsf{a}_2^2| \leq \begin{cases} \mathcal{L}_1(s)\left|\dfrac{1}{12}\right|, & \text{if } \left|\dfrac{(\mathcal{L}_1(s))^2(1-\mu)}{8\left(3(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_2(s) - \mathcal{L}_1(s))\right)}\right| \leq \dfrac{1}{24} \\ 2\mathcal{L}_1(s)\left|\dfrac{(\mathcal{L}_1(s))^2(1-\mu)}{8\left(3(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_2(s) - \mathcal{L}_1(s))\right)}\right|, & \text{if } \left|\dfrac{(\mathcal{L}_1(s))^2(1-\mu)}{8\left(3(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_1(s))^2 - 2(\mathcal{L}_2(s) - \mathcal{L}_1(s))\right)}\right| \geq \dfrac{1}{24} \end{cases}.$$

4. Conclusions

In this paper, we introduced and explored new subclasses of bi-univalent functions $f \in \daleth$ defined through the q-fractional difference operator $\mathcal{J}_{\ltimes}^{\mathsf{q}}$ and linked with the q-Hermite polynomials $\mathcal{L}_k(s \mid \mathsf{q})$. Using subordination, we constructed two main families: the starlike class $\mathcal{B}_{\daleth}(s,\mathsf{q},\ltimes)$ and the convex class $\mathcal{T}_{\daleth}(s,\mathsf{q},\ltimes)$. For functions in these classes, we obtained explicit bounds for the initial coefficients $|a_2|$ and $|a_3|$, and established corresponding Fekete–Szegö inequalities involving the parameters q, s, and $\ltimes$. We also presented several special cases to show how the results simplify to known outcomes from the literature when the parameters take specific values. Overall, our study provides a unified framework connecting q-calculus, fractional operators, and geometric function theory, which can serve as a starting point for further investigations into coefficient problems, operator theory, and extensions to other families of analytic functions. Looking ahead, interesting directions include exploring higher-order coefficients a_n, extending these classes to other q-polynomial families, and investigating potential applications in the unit disk $\ltimes$ and related areas in mathematical physics.

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