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Application of parametric technique in solving linear Neutrosophic partial differential-algebraic controlled Systems with index-3

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Abstract

The current study presents and verifies a new method for solving Neutrosophic Differential-algebraic Controlled Systems with index-3. The method uses the critical point of the formulated Neutrosophic Partial Differential-algebraic variational formulation with Neutrosophic consistent initial condition to be solution for the proposed system and via-versa. With help of differentiation index with respect to time, the reduced constrained control problem in the state-space is then obtained. This transforms variational problem from indirect method into direct method by the technique of generalized Ritz bases. Finally, battery model numerical results confirm the results of the theory.

Mathematics Subject Classification (2010): 34A09, 49J15

Key words and phrases: Partial differential-algebraic equation, consistent initial condition equation, variational formulation.

1. Introduction

Differential-algebraic systems combine differential equation with algebraic equations. Finding an efficient technique to solve differential-algebraic equations is interesting for mathematicians (mathematises) and engineering [1–3].

Many researchers have investigated different differential-algebraic problems in electrical networks [4,5], optimal control systems [6–8], chemical processes problems [9,10] and constrained multibody mechanical systems [11,12]. Differential-algebraic equations with index higher than one can not be solved. Index reduction techniques convert them into lower index problem.

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Numerical methods by different methods to solve higher index of bigger than one [13,14]. They are according the index reduction approaches transforming the higher index Differential-algebraic systems into a lower index decreased one [15,16]. One of index reduction approaches is the differentiation index representing the fewest algebraic equations differentiation times to gain an equivalent reduced pure differential equation [17–19]. Owing to numerical complexities, the method its own is not sufficient to be applied.

The method according to the variational formulation theory by finding suitable functional whose critical points signify the solution of differential-algebraic system. This approach depends firstly on applying method of calulus of variation for the approximation of the analytical solution, then linear combination of the elementary basics of the given separable Banach space and augmented in the resulting variational formulations for parametrization of the solution.

Section 2 is the problem formulation appears. Section 3 gives the reduction technique of the semi explicit Neutrosophic Partial Differential-algebraic Systems with index-3. Sections 4 and 5 show the parametric technique for solving the reduced Neutrosophic Differential-algebraic Controlled Systems. Finally, a practical illustration for this work presented in section 6.

2. Problem Formulation of Linear Neutrosophic Partial differential-algebraic Controlled system with index-3

We discuss in this section solvability of linear neutrosophic partial differential-algebraic Controlled system with index 3. Consider the semi explicit problem

$$\frac{\partial w_1}{\partial t}(t+zI)=A_{11}(t+zI)w_1(t+zI)+A_{12}(t+zI)w_2(t+zI)+B_1(t+zI)u(t)+g_1(t) \tag{1}$$

$$0=A_{21}(t+zI)w_1(t+zI)+B_2(t+zI)u(t)+g_2(t) \tag{2}$$

With $u(t)\in\Delta_u \triangleq$ the admissible control class according the given problem and this leads to uniqueness of our system solution and $A_{ij}, i,j = 1,2$ are matrices with appropriate dimension and A_{ij} is not required to be invertible or square $u\in C^2\left(I,\mathbb{C}^r\right), g_1\in C^1\left(I,\mathbb{C}^{n-n_0}\right),\ g_2\in C^2\left(I,\mathbb{C}^{n-n_0}\right)$ where $I=[t_0,t_f]$.

3. State-Space Reduction of the Semi Explicit Neutrosophic Differential-algebraic Systems with index-3 on Manifold

Recall the problem (1), (2) in section 2 and by using differentiation index of the fewest differentiations for algebraic part (2) for obtaining w_2 as a function continuous of w_1, $\dot{u}$ and $\ddot{u}$.

Now by repeat differentiation for two time to the algebraic equation (2), we get

$$0=\alpha w_1+\beta w_2+\gamma u+\left(A_{21}B_1+\frac{\partial B_2}{\partial t}\right)\dot{u}+B_2\ddot{u}+\delta g_1+A_{21}\dot{g}_1+\ddot{g}_2=F\left(w_1(t+zI)\right) \tag{3}$$

With

$$\alpha=\frac{\partial^2 A_{21}}{\partial t^2}\ddot{A}_{21}+2\frac{\partial A_{12}}{\partial t}A_{11}+A_{21}\frac{\partial A_{11}}{\partial t}+A_{21}A_{11}A_{11} \tag{4}$$

$$\beta=\frac{\partial A_{21}}{\partial t}A_{12}+A_{21}A_{11}A_{12}, \tag{5}$$

$$\gamma=2\frac{\partial A_{21}}{\partial t}B_1+A_{21}\frac{\partial B_1}{\partial t}+A_{21}A_{11}B_1+B_2+\ddot{B}_2 \tag{6}$$

$$\delta = 2\frac{\partial A_{21}}{\partial t} + A_{21}A_{11} \tag{7}$$

If we assume that β is invertable matrix, then one can get

$$\begin{aligned} w_2(t+zI) &= \beta^{-1}\left(-\alpha w_1 - \gamma u - \left(A_{21}B_1 + \frac{\partial B_2}{\partial t}\right)\dot{u} - B_2\ddot{u} - \delta g_1 - A_{21}\dot{g}_1 - \ddot{g}_2\right) \\ &= F(w_2(t+zI), u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2) \end{aligned} \tag{8}$$

The Neutrosophic Differential-algebraic Systems (1)–(2) is decreased to the next controlled system on manifolds

$$\begin{aligned} &\frac{\partial w_1}{\partial t}(t+zI) + \left(A_{12}\beta^{-1}\alpha - A_{11}\right)w_1 \\ &\quad = A_{21}\beta^{-1}\left[-\gamma u - \left(A_{21}B_1 + \dot{B}_2\right)\dot{u} - B_2\ddot{u} - \delta g_1 - A_{21}\dot{g}_1 - \ddot{g}_2\right] + B_1 u + g_1(t) \end{aligned} \tag{9}$$

$$A_{21}(t+zI)w_1(t+zI) = -B_2(t+zI)u(t) - g_2(t) \tag{10}$$

$$\frac{\partial A_{21}}{\partial t} - A_{21}A_{11})w_1(t+zI) = (A_{21}B_1)u(t) + A_{21}g_1(t) + \dot{B}_2 u + B_2\dot{u} + \dot{g}_2(t) \tag{11}$$

One can define class of Neutrosophic consistent initial condition (N.C.I.C.) as

$$k_0 = \left\{\begin{array}{l} w_1(t_0+zI) \in \mathbb{C}^{rank(\alpha)} \ where \\ \text{1) } A_{21}(t_0+zI)w_1(t_0+zI) = -B_2(t_0+zI)u(t_0) - g_2(t_0) \\ \text{2) } (\dot{A}_{21} - A_{21}A_{11})w_1(t_0+zI) = (A_{21}B_1)u(t_0) + A_{21}g_1(t_0) + \dot{B}_2 u + B_2\dot{u} + \dot{g}_2(t_0) \\ \text{satisfied for given } g_1(t_0), g_2(t_0), u(t_0), \dot{u}(t_0) \end{array}\right\}. \tag{12}$$

4. Parametric Technique for Solving the Reduced Neutrosophic Differential-Algebraic Controlled Systems

In this section we will redefine the problem (9–11) as follows

$$L_1 w_1(t+zI) = G_1(u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2) \tag{13}$$

$$L_2 w_1(t+zI) = G_2(u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2) \tag{14}$$

$$L_3 w_1(t+zI) = G_3(u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2) \tag{15}$$

By assuming

$$L_1 w_1(t+zI) \triangleq \frac{d}{dt}w_1(t+zI) + \left(A_{12}\beta^{-1}\alpha - A_{11}\right)w_1(t+zI) \tag{16}$$

$$L_2 w_1(t+zI) \triangleq A_{21}w_1(t+zI) \tag{17}$$

$$L_3 w_1(t+zI) \triangleq \left(\frac{\partial A_{21}}{\partial t} - A_{21}A_{11}\right)w_1(t+zI) \tag{18}$$

and

$$G_1 \triangleq A_{21}\beta^{-1}\left[-\gamma u-\left(A_{21}B_1+\dot{B}_2\right)\dot{u}-B_2\ddot{u}-\delta g_1-A_{21}\dot{g}_1-\ddot{g}_2\right]+B_1u+g_1\left(t\right) \tag{19}$$

$$G_2 \triangleq -B_2\left(t+zI\right)u\left(t\right)-g_2\left(t\right) \tag{20}$$

$$G_3 \triangleq \left(A_{21}B_1\right)u\left(t\right)+A_{21}g_1\left(t\right)+\dot{B}_2u+B_2\dot{u}+\dot{g}_2\left(t\right) \tag{21}$$

Then system (9–11) becomes

$$Lw_1\left(t+zI\right)=\left[L_1,L_2,L_3\right]^T w_1=\left[G_1,G_2,G_3\right]^T=G\left(u,\dot{u},g_1,\dot{g}_1,\ddot{g}_2\right) \tag{22}$$

And

$$Domain\left(L\right)=Dom\left(L\right)=\left\{\left(w,u\right)\in C^1\left(I,\mathbb{C}^{n_0}\right)\times C^2\left(I,\mathbb{C}^r\right) with\, g_1\in C^1\left(I,\mathbb{C}^{n-n_0}\right),\ g_2\in C^2\left(I,\mathbb{C}^{n-n_0}\right)\right\}$$

Since the operator L is linear operator in optimal control problems, then the norm on $C^1(I,\mathbb{C}^{n_0})$ is the sequence's maximum norms on compacting interval I.

It is clear that the operator $Lw_1(t + zI)$ is not symmetric with the produced normal bilinear form $w_1,w_2=\int_{t_0}^{t_f}w_1^Tw_2dt$, see [20–23] Since the operator $\frac{d}{dt}$ appear in $L_1w_1\left(t+zI\right)$ which it leads to the inability to find a variational formulation, and to overcome this limitation, a novel bilinear form must be define.

For creating a variational equal to a linear problem $Lu = G$:

$$J\left[w_1\right]=\int_{t_0}^{t_f}\left[\frac{1}{2}L_1^Tw_1.L_1w_1+\frac{1}{2}L_2^Tw_1.L_2w_1+\frac{1}{2}L_3^Tw_1.L_3w_1-G_1^TL_1w_1-G_2^TL_2w_1-G_3^TL_3w_1\right]dt \tag{23}$$

In practice, locating the critical point of the functional $J[w_1]$ requires the evaluation of it's functional.

However, because this solves the necessary Euler equation corresponding to the problem–challenging problems in its own right-direct methods of variational problem gave an approximate solution in a finite basis number of functions of separable Banach (Hilbert) space of supermum norms.

Then, the parametrization is:

$$w_1^i=w_{10}+\sum_{j=1}^{m_i}a_j^iF_j^i\left(t+zI\right)\ i=1,\ldots,n_0,m\text{ arbitrary}, \tag{24}$$

$$w_2^k=F\left(w_1^i\left(t+zI\right),u,\dot{u},g_1,\dot{g}_1,\ddot{g}_2\right),k=1,\ldots,n-n_0 \tag{25}$$

With

1) $w_{10}\in k_0$
2) F_j^i is time linearly independent basis t.
3) total number of variables $=n=n_0+n-n_0$

Now by substituting (24) and (25) in (23) one can get

$$J\left[w_1\right]=J\left(a_0^1,a_1^1,\ldots,a_{m_1}^1,a_0^2,\ldots,a_{m_2}^2,\ldots,a_0^{n_0},\ldots,a_{m_{n_0}}^{n_0}\right) \tag{26}$$

Setting the derivative of (26) of $a_j^i,i=1,\ldots,n_0$ and $j=1,\ldots,m_i$ to zero is constitutes the critical in variational (23). i.e.

$$\frac{\partial J}{\partial a_j^i}=0,\forall i=1,\ldots,n_0\text{ and }j=1,\ldots,m_i,a_j^i\text{ unknown variables} \tag{27}$$

1) The quadratic structure of the variational formulation allows for the linear system of Neutrosophic algebraic equation to be determined directly from equation (27) with N.C.I.C.

2) Solvability of the system (27) for a_j^i leads to obtain the approximate solution w_1, w_2 according to the equations (24), (25) and so the original solutions (1), (2) gives.

5. A Step-by-Step Algorithm for solving Neutrosophic Differential-algebraic Controlled System

Recall the Neutrosophic semi-explicit system (1–2)

Step 1: Using differentiation index, reduce the system (1)–(2) into equivalent controlled system on manifold

$$\frac{\partial w_1}{\partial t}(t+zI)+\left(A_{12}\beta^{-1}\alpha - A_{11}\right)w_1 = A_{21}\beta^{-1}\left[-\gamma u-\left(A_{21}B_1+\dot{B}_2\right)\dot{u}-B_2\ddot{u}-\delta g_1 - A_{21}\dot{g}_1-\ddot{g}_2\right]+B_1u+g_1(t)$$

$$A_{21}(t+zI)w_1(t+zI) = -B_2(t+zI)u(t)-g_2(t)$$

$$\left(\frac{\partial A_{21}}{\partial t}-A_{21}A_{11}\right)w_1(t+zI) = \left(A_{21}B_1\right)u(t)+A_{21}g_1(t)+\dot{B}_2u+B_2\dot{u}+\dot{g}_2(t)$$

Step 2: Computing the space of N.C.I.C.

$$k_0 = \left\{\begin{array}{l} w_1(t_0+zI)\in\mathbb{C}^{rank(\alpha)}\ where \\ 1)\ A_{21}(t_0+zI)w_1(t_0+zI) = -B_2(t_0+zI)u(t_0)-g_2(t_0) \\ 2)\left(\dfrac{\partial A_{21}}{\partial t}\right)-A_{21}A_{11})w_1(t_0+zI) = (A_{21}B_1)u(t_0)+A_{21}g_1(t_0)+\left(\dfrac{\partial B_2}{\partial t}\right)+B_2\dot{u}+\dot{g}_2(t_0) \\ \text{satisfied for given } g_1(t_0), g_2(t_0), u(t_0), \dot{u}(t_0)) \end{array}\right\}.$$

Step 3: Redefine the problem in step 1 in term of linear operator L

$$Lw_1(t+zI) = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} w_1$$

Where

$$L_1w_1(t+zI) \triangleq \frac{d}{dt}w_1(t+zI)+\left(A_{12}\beta^{-1}\alpha - A_{11}\right)w_1(t+zI)$$

$$L_2w_1(t+zI) \triangleq A_{21}w_1(t+zI)$$

$$L_3w_1(t+zI) \triangleq \left(\frac{\partial A_{21}}{\partial t}-A_{21}A_{11}\right)w_1(t+zI)$$

With

$$Dom(L) = \left\{(w,u)\in C^1\left(I,\mathbb{C}^{n_0}\right)\times C^2\left(I,\mathbb{C}^r\right) \text{ with } g_1\in C^1\left(I,\mathbb{C}^{n-n_0}\right),\ g_2\in C^2\left(I,\mathbb{C}^{n-n_0}\right)\right\}$$

Step 4: Define

$$J\left[w_1\right] = \int_{t_0}^{t_f}\left[\frac{1}{2}L_1^Tw_1\cdot L_1w_1+\frac{1}{2}L_2^Tw_1\cdot L_2w_1+\frac{1}{2}L_3^Tw_1\cdot L_3w_1 - G_1^TL_1w_1 - G_2^TL_2w_1 - G_3^TL_3w_1\right]dt$$

The critical points of $J[w_1]$ solve system in step 1 over $Dom(L)$ with $\cdot,\cdot$ is nondegenerate.

Step 5: Since Banach space can be separated in the solvable space $C\left(\mathbb{C}^{n_0}\right)$, w_1, w_2 as basis function linear combination:

$$w_1^i = w_{10} + \sum_{j=1}^{m_i} a_j^i F_j^i\left(t + zI\right),\ i = 1,\dots,n_0, m\, arbitrary,$$

$$w_2^k = F\left(w_1^i\left(t + zI\right), u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2\right), k = 1,\dots,n - n_0$$

With F_j^i is linearly independent basis function of time t

Step 6: Equate the derivative of the functional $J\left[w_1\right] = J\left(a_0^1, a_1^1, \dots, a_{m_1}^1, a_0^2, \dots, a_{m_2}^2, \dots, a_0^{n_0}, \dots, a_{m_{n_0}}^{n_0}\right)$ to zero is found the critical point of variational formulation $Lu = G$ and an approximate solution to Neutrosophic Partial Differential-algebraic Controlled System is then obtained.

6. Practical Illustration of the Proposed Method

The circuit equations of a non-linear RLC circuit can be considered, under certain conditions, as an implicit description of an ODE on a submanifold of branch voltages and branch currents. So problems in circuit theory, which give rise to higher index formulations are discussed here in the following illustration, see Fig. 1. Since there is no numerical solution exists anywhere, then we solved the system using the proposed approach.

In this series electrical circuit, a single current flows through all components, while the volage is divided across them

$$w(t) = \begin{bmatrix} w_{11} & w_{12} & w_{21} \end{bmatrix} = \begin{bmatrix} \mathcal{V}_1 & \mathcal{V}_2 & I \end{bmatrix}$$

Where $\mathcal{V}_1, \mathcal{V}_2$ are voltages in capacities and I represents amperage on the currents flowing over the capacities [23]

$$\dot{w}_{11} = -w_{12} - w_{21} + u + t^5 + t^3 - 2 + 3t - sint$$

$$\dot{w}_{12} = w_{12} - w_{21} + 5t^4 + 3t^2 - t^3 - t^5 + t - 2$$

$$0 = w_{11} - w_{12} + u + t^5 + t^3 - t^2 - 1 - sint$$

Step 1: Using two time repeated differentiation in terms time to get w_{21}

$$w_2 = -\alpha w_1 - G\left(u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2\right)$$

With $\alpha = A_{12}\left(A_{21}A_{11}A_{12}\right)^T A_{21}A_{11}A_{11} - A_{11}$

And

$$G\left(u, \dot{u}, g_1, \dot{g}_1, \ddot{g}_2\right) = A_{12}\left(A_{21}A_{11}A_{12}\right)^T\left[-A_{21}A_{11}B_1u - A_{21}B_1\dot{u} - \ddot{u} - A_{21}A_{11}g_1 - \dot{g}_1 - \ddot{g}_2\right] + B_1u + g_1$$

Where

$$A_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}; A_{21} = \begin{bmatrix} 1 & -1 \end{bmatrix}; A_{12} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}; B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{12} \\ w_{21} \end{bmatrix}$$

Step 2: Define the reduced system

$$\dot{w}_1 + \left(\alpha - A_{11}\right) w_1 = B_1 u + g_1 - G$$

$$-A_{21} w_1 = u + g_2$$

$$-A_{21} A_{11} w_1 = A_{21} B_1 u + A_{21} g_1 + \dot{u} + \dot{g}_2$$

Step 3: the class of N.C.I.C.

$$k_0 = \left\{ \begin{array}{c} w_1\left(t_0\right) \mid -A_{21} w_1\left(t_0\right) = u\left(t_0\right) + g_2\left(t_0\right) \text{ and} \\ -A_{21} A_{11} w_1\left(t_0\right) = A_{21} B_1 u\left(t_0\right) + A_{21} g_1\left(t_0\right) + \dot{u}\left(t_0\right) + \dot{g}_2\left(t_0\right) \end{array} \right\}$$

$$= \left\{ \left(w_{11}\left(t_0\right), w_{12}\left(t_0\right)\right) \mid w_{21}\left(t_0\right) = 0 \text{ and } - w_{11}\left(t_0\right) + w_{12}\left(t_0\right) = -1 \right\}$$

Step 4: Define the varitional formulation as follows:

$$J[w_1] = \int_{t_0}^{t_f} \left[\begin{array}{l} \frac{1}{2}[\dot{w}_1 + (\alpha - A_{11}) w_1]^T [\dot{w}_1 + (\alpha - A_{11}) w_1] + \frac{1}{2}[A_{21} w_1]^T [A_{21} w_1] \\ + \frac{1}{2}[A_{21} A_{11} w_1]^T [A_{21} A_{11} w_1] - [B_1 u + g_1 - G]^T [\dot{w}_1 + (\alpha - A_{11}) w_1] \\ - [u + g_2]^T [A_{21} w_1] - [A_{21} B_1 u + A_{21} g_1 + \dot{u} + \dot{g}_2]^T [A_{21} A_{11} w_1] dt \end{array} \right]$$

Step 5: Set $w_{11} = w_{10} + \sum_{j=1}^{5} a_{j1} t^j ; w_{12} = w_{20} + \sum_{j=1}^{5} a_{j2} t^j ;$

Step 6: Find the approximate solution (w_{11}, w_{12}, w_{21}) by setting $\frac{\partial J}{\partial a_{j1}} = 0$ and $\frac{\partial J}{\partial a_{j2}} = 0; j = 1,..,5$, and this leads to algebraic equation $x \begin{pmatrix} a_{j1} \\ a_{j2} \end{pmatrix} = y$, which solvable for $\begin{pmatrix} a_{j1} \\ a_{j2} \end{pmatrix} = x^{-1} y$.

Step 7: the numerical results given by:

a_{01}	1	a_{31}	-1×10^{-39}
a_{02}	0	a_{32}	1
a_{11}	3×10^{-41}	a_{41}	2×10^{-39}
a_{12}	-1×10^{-39}	a_{42}	3×10^{-38}
a_{21}	1	a_{51}	-1×10^{-39}
a_{22}	-1×10^{-38}	a_{52}	1

The approximate solution of the differential-algebraic states are shown in figures 2, 3, 4 respectively.

8. Conclusion

An efficient approach of solving a higher index of neutrosophic partial differential-algebraic controlled system is presented in this paper. The approach uses the differentiation index with respect to time and implicit function theorems. The differentiation index is repeated until the required circumstances by the implicit function theorems for solving the algebraic equations for the unknown state-space parameters for forming a decreased neutrosophic normal differential equation on algebraic limits with neutrosophic confirming initial conditions.

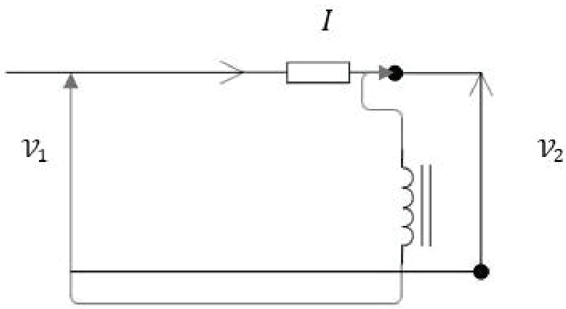

Figure 1: Electrical circuit.

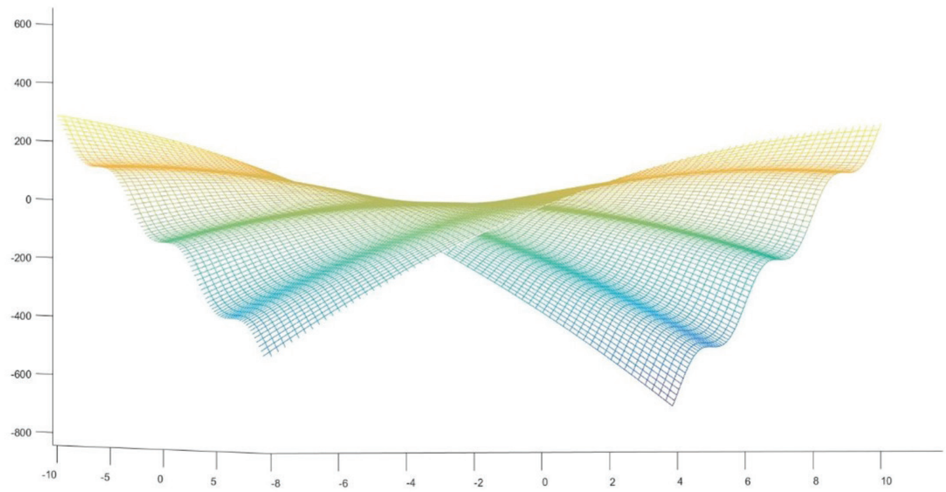

Figure 2: Neutrosophic differential state w_{11} with N.C.I.C. k_0.

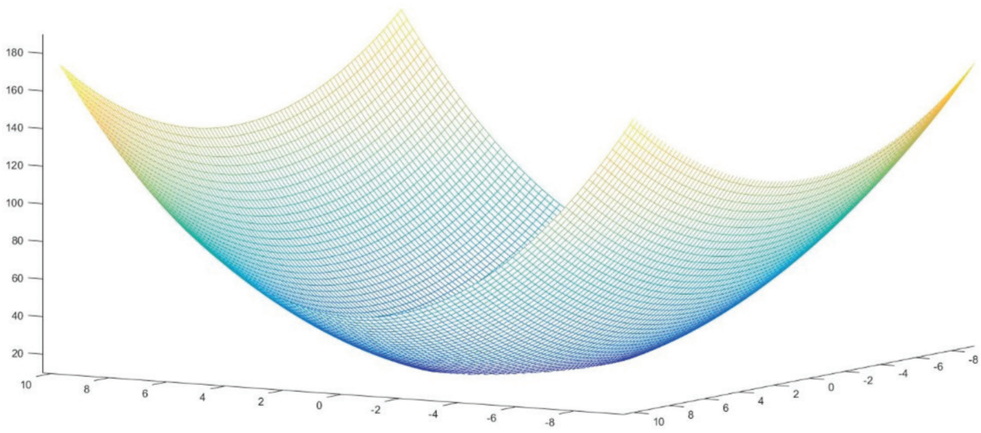

Figure 3: Neutrosophic differential state w_{12} with N.C.I.C. k_0.

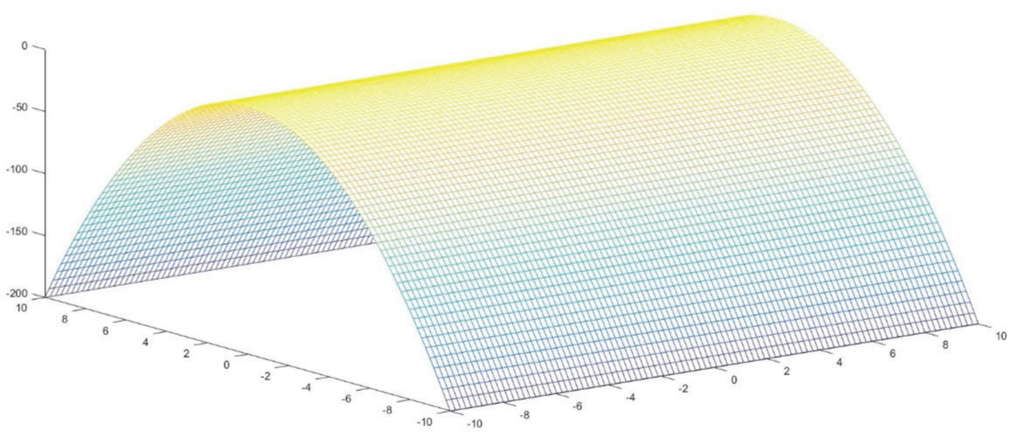

Figure 4: Equality state w_{21} with N.C.I.C. k_0.

in an operator form, the system is written with a developed variational formulation. The solution is then linearly combining some the setting space elements. So, the efficient approximate solutions are produced even for a small number of polynomial basis function.

Future works

1. There is a plan to investigate the solvability of neutrosophic differential-algebraic controlled System with index higher than 3 (index-k as generalization).
2. The stability of neutrosophic differential-algebraic controlled system under neutrosophic consistent initial conditions could be investigated in future studies.

We therefore feel that our approach could be made quite effective from more complicated problems.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

There is no interest conflict.

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