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Applying fixed point theorems for generalized kannan-type mappings to fredholm equations in the setting of extended *b*-metric spaces

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Abstract

This paper studies a comprehensive study of fixed point theory in extended b-metric spaces and its applications to Fredholm integral equations, with a particular focus on Hyers–Ulam stability. We provide new fixed point theorems for extended Kannan-type mapping. using subadditive altering distance functions, significantly extending classical results from standard metric and b-metric spaces to this more flexible and generalized framework .

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Key words and Phrases: Fixed point (FP), Extended b-metric space (EBMS), Kannan type mapping (KTM), Hyers-Ulam Stability (HUS), Fredholm integral equation (FIE).

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1. Introduction and Preliminary

Fixed point theory stands as one of the most widely applied branches of mathematical analysis, with its foundations rooted in the seminal Banach contraction principle [4]. This elegant theory has evolved significantly through various generalizations, most notably through Kannan's groundbreaking work [13], which introduced discontinuous mappings that still guarantee fixed points and characterize metric completeness.

The subsequent development of b-metric spaces (BMS) by Bakhtin [3] and Czerwik [5] further expanded this framework by relaxing the triangle inequality through a constant coefficient $s \geq 1$. The

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natural progression of this research has led to the introduction of extended b-metric spaces (EBMS), where the constant coefficient s is replaced by a function $s : X \times X \to [1, \infty)$. This generalization represents a significant advancement in metric space theory, offering greater flexibility and applicability to complex mathematical structures that arise in various scientific and engineering contexts. The variable coefficient function allows for localized control of the triangle inequality, making EBMS particularly suitable for modeling nonlinear phenomena and singular behaviors that cannot be adequately described by classical metric spaces.

Parallel to these spatial generalizations, the concept of stability in functional equations has emerged as a crucial research area, with Hyers-Ulam stability (HUS) occupying a central position. This stability concept addresses a fundamental question in mathematical analysis: if an equation is approximately satisfied, is there an exact solution nearby? The importance of HUS extends beyond pure mathematics, finding applications in numerical analysis, approximation theory, and mathematical modeling where exact solutions may be unattainable but approximate solutions are available.

Integral equations, particularly Fredholm integral equations (FIE), represent one of the most important application areas of fixed point theory. These equations arise naturally in various physical, engineering, and economic contexts, including heat transfer, population dynamics, signal processing, and optimal control problems. The examination of existence, uniqueness, and stability of solutions to integral equations has been a central theme in applied analysis for over a century.

Theorem 1.1. *Let (X, d, s) be a CEBMS with $S = \sup\{s(x, y) : x, y \in X\}$, where $1 \le S < \infty$. Let $T : X \to X$ be a mapping such that there exists a constant $\kappa \in \left(0, \frac{1}{2S}\right)$ satisfying*

$$d(Tx,Ty) \le \kappa\{d(x,Tx) + d(y,Ty)\} \qquad \text{for all } x,y \in X \tag{1.1}$$

Then T has a unique fixed point $\varpi \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to ϖ.

The relevance of the above conclusion stems from the Kannan theorem (KT), which describes the completeness of metric spaces. This was shown by Subrahmanyam [15] in 1975. Theorem 1.1 is one of the first modifications of the Banach contraction principle, obtained by either altering the contraction condition or generalizing the space (see, for example, [6–8, 15, 17, 18], among others). In 1989, Bakhtin [3] introduced extended b-metric spaces (EBMS) to extend the Banach fixed-point theory. In 1993, Czerwik [5] established b-metric spaces (BMS) as follows.

Definition 1.1. [10] Let X be a non-empty set and let $s \ge 1$ be a real number. A mapping $d : X \times X \to \mathbb{R}$ is called a b-metric on X if it satisfies the following conditions for all $x, y, z \in X$:

1. $d(x, y) \ge 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \le s\,[d(x, z) + d(z, y)$.

Then the pair (X, d) is called a b-metric space (BMS). If $s = 1$, then the b-metric space reduces to a standard metric space (MS).

Consequently, many authors have extended the Banach fixed point theorem to b-metric spaces (see, for example, [1, 12, 13, 14, 16] and the references therein).

Definition 1.2. Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ and $s : X \times X \to [1,\infty)$ be functions satisfying for all $x, y, z \in X$:

1. $d(x, y) \ge 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \le s\,[d(x, z) + d(z, y)$.

Then *(X, d, s)* is called an extended b-metric space (EBMS). If, in addition, *(X, d, s)* is complete, we call it a complete extended b-metric space (CEBMS).

Remark 1.2. In the classical definition of extended b-metric spaces, the function $s : X \times X \to [1, \infty)$ need not be bounded. However, throughout this paper, we consider only bounded extended b-metric spaces, i.e.,

$$S = \sup\{s(x,y) : x, y \in X\} < \infty, \quad \text{with } 1 \le S < \infty.$$

This assumption guarantees that all constants and series appearing in the proofs are well-defined and convergent. All fixed point results established in this work are therefore valid for this subclass of EBMS.

Definition 1.3. A function $\vartheta : [0,\infty) \to [0,\infty)$ is called an altering distance function if it satisfies the following conditions:

1. ϑ is continuous and strictly increasing,
2. $\vartheta(t) = 0$ if and only if $t = 0$,
3. ϑ is subadditive, i.e., $\vartheta(a+b) \le \vartheta(a) + \vartheta(b), \quad \forall a,b \ge 0$,
4. ϑ is invertible on its range.

Example 1.3. The following easily seen that function:

1. $\vartheta_1(\zeta) = \gamma\zeta \; \forall \gamma \ge 1$
2. $\vartheta_2(\zeta) = \sqrt[n]{\zeta}, n \in N$
3. $\vartheta_3(\zeta) = \log(1+\zeta), \zeta \ge 0$
4. $\vartheta_4(\zeta) = \tan^{-1}\zeta$

Are sub-additive distance functions altered

2. Principal Findings

Theorem 2.1. *Let (X, d, s) be a complete EBMS and let T: X → X be a mapping such that for some* $p < \dfrac{1}{2\sup s + 1}$ *the following holds*

$$\vartheta(d(Tx,Ty)) \le p\{\vartheta(d(x,y)) + \vartheta(d(x,Tx)) + \vartheta(d(y,Ty))\} \tag{2.1}$$

for all $x, y \in X$. *Then T has a unique fixed point* $x^* \in X$ *and for every* $x_0 \in X$, *the iterates of the sequence* $\{T^n x_0\}$ *converge to x*.*

Proof. Let $S = \sup\{s(x,y) : x, y \in X\}$ where $1 \le S < \infty$. Therefore $p < \dfrac{1}{2S+1}$.

For any $x \in X$, let $y = Tx$. Then

$$\begin{aligned}\vartheta(d(y,Ty)) &\le p\{\vartheta(d(x,y)) + \vartheta(d(x,Tx)) + \vartheta(d(y,Ty))\} \\ &\le p\vartheta(d(x,y)) + p\vartheta(d(x,Tx)) + p\vartheta(d(y,Ty)).\end{aligned}$$

Thus

$$\vartheta(d(y,Ty))(1-p) \le 2p\vartheta(d(x,y)).$$

Hence

$$\vartheta(d(y,Ty)) \le \frac{2p}{1-p}\vartheta(d(x,y)) = q\vartheta(d(x,y)),$$

where $q = \dfrac{2p}{1-p}$.

Since ϑ is strictly increasing and invertible (see Definition 1.3), we apply ϑ^{-1} to both sides to obtain

$$d(y,Ty) \le \vartheta^{-1}(q\vartheta(d(x,y))).$$

In the special case where $\vartheta(t) = t$, we have $\vartheta(t)^{-1} = t$, and the inequality reduces to

$$d(y,Ty) \le q d(x,y).$$

Throughout the main theorem, we work with $\vartheta(t) = t$. Thus, for any $x_0 \in X$, consider the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$. Then by induction,

$$d(x_n, x_{n+1}) \le q^n d(x_0, x_1).$$

For any $n, m \in \mathbb{N}$ with $m > n$, using the EBMS inequality:

$$\begin{aligned}
d(x_n,x_m) &\le s(x_n,x_m)[d(x_n,x_{n+1}) + d(x_{n+1},x_m)] \\
&\le S[d(x_n,x_{n+1}) + d(x_{n+1},x_m)] \\
&\le Sd(x_n,x_{n+1}) + d(x_{n+1},x_m) \\
&\le Sd(x_n,x_{n+1}) + S^2[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)] \\
&\le Sd(x_n,x_{n+1}) + S^2 d(x_{n+1},x_{n+2}) + S^2 d(x_{n+2},x_m) \\
&\le Sd(x_n,x_{n+1}) + S^2 d(x_{n+1},x_{n+2}) + S^3 d(x_{n+2},x_{n+3}) + \cdots + S^{m-n+1} d(x_{m-1},x_m) \\
&\le Sq^n d(x_0,x_1) + S^2 q^{n+1} d(x_0,x_1) + \cdots + S^{m-n} q^{m-1} d(x_0,x_1) \\
&\le q^n d(x_0,x_1)(S + S^2 q + \cdots + S^{m-n} q^{m-n-1}).
\end{aligned}$$

Since $q = \dfrac{2p}{1-p}$ and $p < \dfrac{1}{2S+1}$, we have $Sq < 1$. Therefore the series converges and

$$d(x_n,x_m) \le q^n d(x_0,x_1) \cdot \frac{S}{1-qS} \to 0 \quad \text{as } n \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence in the CEBMS (X, d, s).

Now we show that x^* is a fixed point. Using the EBMS inequality and the contraction condition 2.1

$$\begin{aligned}
\vartheta(d(Tx^*,x^*)) &\le \vartheta(s(Tx^*,x^*)[d(Tx^*,Tx_n) + d(Tx_n,x^*)]) \\
&\le S[\vartheta(d(Tx^*,Tx_n)) + \vartheta(d(Tx_n,x^*))] \\
&\le Sq(\vartheta(d(x^*,x_n)) + \vartheta(d(x^*,Tx^*)) + \vartheta(d(x_n,Tx_n))) + S\vartheta(d(x_{n+1},x^*))
\end{aligned}$$

Hence

$$\vartheta(d(x^*,Tx^*))(1-Sq) \le Sq\vartheta(d(x^*,x_n)) + Sq\vartheta(d(x_n,Tx_n)) + S\vartheta(d(x_{n+1},x^*)).$$

As $n \to \infty$, we have

$$\vartheta(d(x^*,Tx^*))(1-Sq) = 0.$$

Since $1 - Sq > 0$, it follows that $\vartheta(d(x^*,Tx^*)) = 0$. By Definition 1.3, since $\vartheta(t) = 0$ if and only if $t = 0$, we obtain $d(x^*, Tx^*) = 0$, and hence $Tx^* = x^*$.

Now we show uniqueness. Suppose that $\mu \in X$ is another fixed point of T. Then

$$\begin{aligned}\vartheta(d(x^*,\mu)) &= \vartheta(d(Tx^*,T\mu)) \\ &\le p\{\vartheta(d(x^*,\mu)) + \vartheta(d(x^*,Tx^*)) + \vartheta(d(\mu,T\mu))\} \\ &\le p\{\vartheta(d(x^*,\mu)) + \vartheta(d(x^*,x^*)) + \vartheta(d(\mu,\mu))\} \\ &= p\vartheta(d(x^*,\mu)).\end{aligned}$$

Thus $\vartheta(d(x^*,\mu))(1-p) \le 0$. Since $p < 1$, we have $\vartheta(d(x^*,\mu)) = 0$. By Definition 1.3 $\vartheta(t) = 0$ if and only if $t = 0$, $d(x^*,\mu) = 0$, so $x^* = \mu$. Finally, we obtain the error estimate

$$\begin{aligned}d(T^{n+1}x_0,T^n x_0) &\le q d(T^{n-1}x_0,T^n x_0) \\ &\le q^n d(x_0,Tx_0),\end{aligned}$$

where $q = \dfrac{2p}{1-p} < 1.$ ☐

Example 2.2. Let $X = [0, 0.5]$ and define the coefficient function $s : X \times X \to [1, \infty)$ by

$$s(x,y) = 1 + \frac{|x-y|}{2}.$$

Define the extended b-metric $d : X \times X \to [0, \infty)$ by

$$d(x,y) = |x-y|, \qquad \forall x,y \in X.$$

Define the mapping $T: X \to X$ by

$$Tx = \frac{x}{3}, \qquad \forall x \in X.$$

Take the altering distance function $\vartheta(t) = t$ for all $t \ge 0$.

For all $x,y,z \in X$:

$$\begin{aligned}d(x,z) &= |x-z| \\ &\le |x-y| + |y-z| \\ &\le \left(1 + \frac{|x-z|}{2}\right)[\,|x-y| + |y-z|\,] \\ &= s(x,z)[d(x,y) + d(y,z)].\end{aligned}$$

The other metric axioms are clearly satisfied. Since $X = [0, 0.5]$ is closed and bounded in $\mathbb{R}$, and s is continuous, (X, d, s) is a complete extended b-metric space. For all $x,y \in [0, 0.5]$, we have $|x-y| \le 0.5$. Hence

$$s(x,y) = 1 + \frac{|x-y|}{2} \le 1 + \frac{0.5}{2} = 1.25.$$

The supremum is attained at $x = 0.5$, $y = 0$, so

$$S = \sup_{x,y\in X} s(x,y) = 1.25, \quad \text{with } 1 \le S < \infty.$$

Therefore,

$$\frac{1}{2S+1} = \frac{1}{2(1.25)+1} = \frac{1}{3.5} \approx 0.285714.$$

Verification of the Kannan-type condition 2.1. We need to find $p < \dfrac{1}{2S+1}$ such that

$$d(Tx,Ty) \le p\{d(x,y) + d(x,Tx) + d(y,Ty)\}, \qquad \forall x, y \in X.$$

Compute each term:

$$d(Tx,Ty) = \left|\frac{x}{3} - \frac{y}{3}\right| = \frac{|x-y|}{3},$$

$$d(x,Tx) = \left|x - \frac{x}{3}\right| = \frac{2x}{3},$$

$$d(y,Ty) = \left|y - \frac{y}{3}\right| = \frac{2y}{3}.$$

The inequality becomes:

$$\frac{|x-y|}{3} \le p\left(|x-y| + \frac{2x}{3} + \frac{2y}{3}\right), \qquad \forall x, y \in [0, 0.5].$$

Choose $p = 0.25$. Then $p = 0.25 < 0.285714$, so the condition $p < \dfrac{1}{2S+1}$ is satisfied.

Now verify the inequality directly:

$$\frac{|x-y|}{3} \le 0.25\left(|x-y| + \frac{2(x+y)}{3}\right)$$

Multiply both sides by 12.

$$4|x-y| \le 3|x-y| + 2(x+y)$$
$$|x-y| \le 2(x+y).$$

The last inequality holds for all $x, y \ge 0$ because $|x-y| \le x + y \le 2(x+y)$. Hence the contractive condition 2.1 is satisfied for all $x, y \in [0, 0.5]$ with $p = 0.25$.

To find the unique fixed point .

Solving $Tx = x$ gives $\frac{x}{3} = x \Rightarrow x = 0$. Clearly $0 \in X$, so 0 is the unique fixed point of T.

For any initial point $x_0 \in [0, 0.5]$, the Picard sequence is

$$x_n = T^n x_0 = \frac{x_0}{3^n}.$$

Clearly $x_n \to 0$ as $n \to \infty$, confirming convergence to the fixed point.

From the proof of Theorem 2.1, we have

$$d(x_{n+1}, x_n) \le q^n d(x_0, x_1), \quad \text{where } q = \frac{2p}{1-p}.$$

With $p = 0.25$, we get $q = \dfrac{2(0.25)}{1-0.25} = \dfrac{0.5}{0.75} = \dfrac{2}{3} \approx 0.6667 < 1.$

For $x_0 = 0.5$, we have $x_1 = T(0.5) = \dfrac{0.5}{3} \approx 0.1667$, so $d(x_0, x_1) = |0.5 - 0.1667| = 0.3333$. The error estimate gives:

$$d(x_{n+1}, x_n) \le \left(\frac{2}{3}\right)^n \cdot 0.3333.$$

The numerical verification:

n	x_n	$d(x_{n+1},x_n)$	$\left(\frac{2}{3}\right)^n \cdot 0.3333$
0	0.5000	0.3333	0.3333
1	0.1667	0.1111	0.2222
2	0.0556	0.0370	0.1481
3	0.0185	0.0123	0.0988
4	0.0062	0.0041	0.0658
5	0.0021	0.0014	0.0439

The numerical results confirm the theoretical error estimate and convergence to the fixed point 0.

Corollary 2.3. *Let* (X, d, s) *be a CEBMS with* $S = \sup\{s(x,y) : x, y \in X\}$, *where* $1 \le S < \infty$. *Let* $T : X \to X$ *be a mapping such that for some* $p < \frac{1}{2S+1}$, *the following inequality holds for all* $x, y \in X$*:*

$$\ln(1 + d(Tx,Ty)) \le p\{\ln(1 + d(x,y)) + \ln(1 + d(x,Tx)) + \ln(1 + d(y,Ty))\}. \tag{2.2}$$

Then T *has a unique fixed point* $x^* \in X$. *Moreover, for every* $x_0 \in X$, *the Picard sequence* $\{T^n x_0\}$ *converges to* x^*, *and for* $q = \frac{2p}{1-p} < 1$, *the error estimate*

$$d(T^{n+1}x_0, T^n x_0) \le q^n d(x_0, Tx_0), \qquad n = 0,1,2,\ldots \tag{2.2}$$

holds.

Proof. Define $\vartheta : [0,\infty) \to [0,\infty)$ by $\vartheta(t) = \ln(1+t)$. This function satisfies all conditions of Definition 1.3: it is strictly increasing, continuous, satisfies $\vartheta(t) = 0 \Leftrightarrow t = 0$, and is invertible via $\vartheta^{-1}(t) = e^t - 1$. With this choice, inequality (2.2) becomes exactly condition (2.1) of Theorem 2.1. Since $p < \frac{1}{2S+1}$, all hypotheses of Theorem 2.1 are satisfied, and therefore T has a unique fixed point $x^* \in X$, with the Picard sequence converging to x^* and satisfying the error estimate 2.3. ☐

Example 2.4. Consider the CEBMS (X, d, s), where $X = [0, 1]$, the metric is defined by

$$d(x,y) = |x-y|^2, \qquad \forall x, y \in X,$$

and the coefficient function $s : X \times X \to [1, \infty)$ is defined by

$$s(x,y) = 2 + |x-y|, \qquad \forall x, y \in X.$$

For all $x, y, z \in X$:

$$d(x,y) = 0 \Leftrightarrow |x-y|^2 = 0 \Leftrightarrow x = y,$$
$$d(x,y) = |x-y|^2 = d(y,x).$$

For the extended triangle inequality, using $|a+b|^2 \le 2(|a|^2 + |b|^2)$:

$$\begin{aligned} d(x,z) &= |x-z|^2 \\ &\le 2(|x-y|^2 + |y-z|^2) \\ &\le (2 + |x-z|)[d(x,y) + d(y,z)] \\ &= s(x,z)[d(x,y) + d(y,z)], \end{aligned}$$

since $2 \le 2 + |x - z|$ for all $x, z \in [0,1]$.

Since $X = [0, 1]$ is closed and bounded in $\mathbb{R}$, the space (X, d, s) is complete. Note that (X, d) is not a classical metric space, since the triangle inequality does not hold in general. Indeed,

$$d(0,1) = 1, \quad d(0,0.5) + d(0.5,0) = 0.25 + 0.25 = 0.5,$$

so the triangle inequality fails. However, the extended triangle inequality is satisfied via the function $s(x, y)$, and hence (X, d, s) is an extended b-metric space.

For all $x, y \in [0,1]$:

$$|x - y| \le 1,$$

so

$$s(x, y) = 2 + |x - y| \le 2 + 1 = 3.$$

The supremum is attained at $x = 1$, $y = 0$, giving

$$S = \sup\{s(x, y) : x, y \in X\} = 3, \quad 1 \le S < \infty.$$

Therefore

$$\frac{1}{2S+1} = \frac{1}{7} \approx 0.1429.$$

Choose $p = 0.1 < \frac{1}{7}$.

Define $T\colon X \to X$ by

$$Tx = \frac{x}{10}, \qquad \forall x \in X.$$

Since $x \in [0, 1]$ implies $Tx \in [0, 0.1] \subset [0, 1]$, the mapping is well-defined. By Corollary 2.3, we verify:

$$\ln(1 + d(Tx, Ty)) \le p\{\ln(1 + d(x, y)) + \ln(1 + d(x, Tx)) + \ln(1 + d(y, Ty))\}.$$

We compute:

$$d(Tx, Ty) = \frac{|x - y|^2}{100},$$
$$d(x, Tx) = \left(\frac{9}{10}\right)^2 x^2 = \frac{81x^2}{100},$$
$$d(y, Ty) = \frac{81y^2}{100}.$$

Worst case: $x = 1$, $y = 0$:

$$\begin{aligned}
\text{LHS} &= \ln\left(1 + \frac{1}{100}\right) = \ln(1.01) \approx 0.00995,\\
\text{RHS} &= 0.1\left\{\ln(1+1) + \ln\left(1 + \frac{81}{100}\right) + \ln(1+0)\right\}\\
&= 0.1\{\ln 2 + \ln(1.81)\}\\
&= 0.1\{0.6931 + 0.5933\}\\
&= 0.1 \times 1.2864 = 0.12864.
\end{aligned}$$

Since $0.00995 \le 0.12864$, the condition (2.2) holds.

General case: For all $x, y \in [0,1]$, since $|x-y|^2 \le 1$:

$$\ln\left(1+\frac{|x-y|^2}{100}\right) \le \ln(1.01) \approx 0.00995 \le 0.12864 \le \text{RHS}.$$

Hence the Kannan-type condition holds for all $x, y \in X$.
Now to showing unique fixed point and convergence.
Setting $Tx^* = x^*$:

$$\frac{x^*}{10} = x^* \Rightarrow x^* = 0.$$

Since $0 \in [0,1]$, the unique fixed point is $x^* = 0$.
The Picard sequence is:

$$T^n x_0 = \frac{x_0}{10^n} \xrightarrow{n\to\infty} 0.$$

The sequence converges rapidly to $x^* = 0$, confirming all theoretical results of Corollary 2.3.

Theorem 2.5. *Let* (X, d, s) *be a CEBMS with* $S = \sup\{s(x,y) : x, y \in X\}$, *where* $1 \le S < \infty$. *Let* $T : X \to X$ *be a mapping and let* $\vartheta : [0,\infty) \to [0,\infty)$ *be an altering distance function satisfying Definition 1.3. Suppose there exist constants* $p_1, p_2, p_3 \ge 0$ *satisfying*

$$p_1 + p_2 + p_3 < \frac{1}{2S}.$$

In particular, this implies that $S(p_1 + p_2 + p_3) < 1$, *and the following inequality holds for all* $x, y \in X$:

$$\vartheta\big(d(Tx,Ty)\big) \le p_1\vartheta\big(d(x,y)\big) + p_2\vartheta\big(d(x,Tx)\big) + p_3\vartheta\big(d(y,Ty)\big). \tag{2.4}$$

Then T has a unique fixed point $x^* \in X$, *and for every* $x_0 \in X$, *the Picard sequence* $\{T^n x_0\}$ *converges to* x^*. *Moreover, for*

$$q = \frac{p_1 + p_2}{1 - p_3} < 1,$$

the following error estimate holds

$$d(T^{n+1}x_0, T^n x_0) \le q^n d(x_0, Tx_0), \qquad n = 0,1,2,\ldots \tag{2.5}$$

Proof. Let $x \in X$, and $\xi = Tx$. Then

$$\begin{aligned}\vartheta(d(\xi,T\xi)) &= \vartheta(d(Tx,T\xi))\\ &\le p_1\vartheta(d(x,\xi)) + p_2\vartheta(d(x,Tx)) + p_3\vartheta(d(\xi,T\xi))\\ &= p_1\vartheta(d(x,\xi)) + p_2\vartheta(d(x,\xi)) + p_3\vartheta(d(\xi,T\xi))\\ &= (p_1+p_2)\vartheta(d(x,\xi)) + p_3\vartheta(d(\xi,T\xi)).\end{aligned}$$

Thus

$$\vartheta(d(\xi,T\xi)) \le \frac{p_1+p_2}{1-p_3}\vartheta(d(x,\xi)).$$

Since ϑ is strictly increasing and continuous in Definition 1.3, there exists $q < 1$ such that $\vartheta(d(\xi,T\xi)) \le q\vartheta(d(x,\xi))$. Taking $q = \dfrac{p_1+p_2}{1-p_3}$.

For any $x_0 \in X$, define the sequence $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. We have

$$d(x_{n+1}, x_{n+2}) \leq q d(x_n, x_{n+1}).$$

By induction,

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1).$$

For $m > n$, using the EBMS inequality:

$$\begin{aligned} d(x_n, x_m) &\leq s(x, y)[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq S[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq Sd(x_n, x_{n+1}) + S^2 d(x_{n+1}, x_{n+2}) + S^3 d(x_{n+2}, x_{n+3}) + \cdots + S^{m-n} d(x_{m-1}, x_m) \\ &\leq Sq^n d(x_0, x_1) + S^2 q^{n+1} d(x_0, x_1) + \cdots + S^{m-n} q^{m-1} d(x_0, x_1) \\ &= q^n d(x_0, x_1)[S + S^2 q + \cdots + S^{m-n} q^{m-n-1}]. \end{aligned}$$

Since $q = \dfrac{p_1 + p_2}{1 - p_3} < 1$ and $Sp_2 < 1$, we have $Sq < 1$ and $p_1 + p_2 + p_3 < 1$. Thus

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) \cdot \frac{S}{1 - Sq} \to 0 \quad \text{as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in the CEBMS (X, d, s), so there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Now we show that x^* is a fixed point.

$$\begin{aligned} \vartheta(d(Tx^*, x^*)) &\leq \vartheta(s(Tx^*, x^*)[d(Tx^*, Tx_n) + d(Tx_n, x^*)]) \\ &\leq S[\vartheta(d(Tx^*, Tx_n)) + \vartheta(d(Tx_n, x^*))] \\ &\leq S[p_1 \vartheta(d(x^*, x_n)) + p_2 \vartheta(d(x^*, Tx^*)) + p_3 \vartheta(d(x_n, Tx_n))] + S\vartheta(d(x_{n+1}, x^*)). \end{aligned}$$

Thus

$$(1 - Sp_2)\vartheta(d(Tx^*, x^*)) \leq Sp_1 \vartheta(d(x^*, x_n)) + Sp_3 \vartheta(d(x_n, Tx_n)) + S\vartheta(d(x_{n+1}, x^*)) \quad \text{as } n \to \infty.$$

By Definition 1.3, $\vartheta(t) = 0 \to t = 0$. Since $p_1 + p_2 + p_3 < 1$ and $(1 - Sp_2) > 0$, therefore

$$(1 - Sp_2)\vartheta(d(Tx^*, x^*)) = 0 \Rightarrow \vartheta(d(Tx^*, x^*)) = 0 \Rightarrow d(Tx^*, x^*) = 0.$$

Hence $Tx^* = x^*$. Now we show uniqueness. Suppose $\mu \in X$ is another fixed point of T.

$$\begin{aligned} \vartheta(d(x^*, \mu)) &= \vartheta(d(Tx^*, T\mu)) \\ &\leq p_1 \vartheta(d(x^*, \mu)) + p_2 \vartheta(d(x^*, Tx^*)) + p_3 \vartheta(d(\mu, T\mu)) \\ &= p_1 \vartheta(d(x^*, \mu)) \\ (1 - p_1)\vartheta(d(x^*, \mu)) &\leq 0 \end{aligned}$$

Since $p_1 < 1$ and $\vartheta(t) = 0 \Leftrightarrow t = 0$, which implies $\vartheta(d(x^*, \mu)) = 0$, hence $d(x^*, \mu) = 0$. Therefore $x^* = \mu$. □

Theorem 2.3. *Let (X, d, s) be a CEBMS with* $S = \sup\{s(x, y) : x, y \in X\}$, *where* $1 \leq S < \infty$. *Let* $\vartheta : [0, \infty) \to [0, \infty)$ *be an altering distance function satisfying Definition 1.3, and let* $T : X \to X$ *be a mapping such that for some* $0 \leq p < \dfrac{1}{2S + 1}$, *the inequality*

$$\vartheta\big(d(Tx, Ty)\big) \leq p\{\vartheta\big(d(x, y)\big) + \vartheta\big(d(x, Tx)\big) + \vartheta\big(d(y, Ty)\big)\} \tag{2.6}$$

holds for all $x, y \in X$. *Then T has a unique fixed point* $x^* \in X$, *and for every* $x_0 \in X$, *the Picard sequence* $\{T^n x_0\}$ *converges to* x^*.

Proof. The result follows directly from Theorem 2.1.

Theorem 2.4. *Let (X, d, s) be an EBMS with* $S = \sup\{s(x,y) : x, y \in X\}$, $1 \leq S < \infty$. *Let* $\vartheta : [0,\infty) \to [0,\infty)$ *be an altering distance function satisfying Definition 3.1, and let* $T : X \to X$ *be a mapping such that for some* $0 \leq p < \dfrac{1}{2S+1}$, *the inequality*

$$\vartheta\big(d(Tx,Ty)\big) \leq p\{\vartheta\big(d(x,y)\big) + \vartheta\big(d(x,Tx)\big) + \vartheta\big(d(y,Ty)\big)\}$$

holds for all $x, y \in X$. *Then T has a unique fixed point* $x^* \in X$ *and the sequence of iterates* $\{T^n x\}$ *converges to* x^*. *Moreover, for* $q = \dfrac{p}{1-p} < 1$,

$$d(T^{n+1}x, x^*) \leq q^n d(x,Tx), \qquad n = 0,1,2,\ldots$$

Since (X, d) is a complete metric space, by taking $\vartheta(t) = t$ *in Theorem 2.1, we recover Theorem 1.1, namely the classical Kannan fixed point theorem.*

Example 2.8. Let $X = \{a, b, c\}$ be a set with three elements. Define the coefficient function $s : X \times X \to [1,\infty)$ by

$$s(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 3, & \text{if } x \neq y. \end{cases}$$

Define the extended b-metric $d : X \times X \to [0,\infty)$ by

$$d(a,b) = 1, \quad d(a,c) = 3, \quad d(b,c) = 1,$$

and $d(x,x) = 0$, $d(x,y) = d(y,x)$ for all $x, y \in X$.

The space (X, d) is not a metric space: In a metric space, the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ must hold for all $x, y, z \in X$. However, taking $x = a$, $y = b$, $z = c$, we have

$$d(a,c) = 3, \quad d(a,b) + d(b,c) = 1 + 1 = 2,$$

so $3 \leq 2$ is false. Hence the triangle inequality fails, and (X, d) is not a metric space. Consequently, the classical Kannan fixed point theorem cannot be applied.

We verify the extended triangle inequality $d(x,z) \leq s(x,z)[d(x,y) + d(y,z)]$ for all $x, y, z \in X$:

- For $(x, y, z) = (a, b, c)$:

$$d(a,c) = 3 \leq s(a,c)[d(a,b) + d(b,c)] = 3 \times (1+1) = 6.$$

- For $(x, y, z) = (a, c, b)$:

$$d(a,b) = 1 \leq s(a,b)[d(a,c) + d(c,b)] = 3 \times (3+1) = 12.$$

- For $(x, y, z) = (b, a, c)$:

$$d(b,c) = 1 \leq s(b,c)[d(b,a) + d(a,c)] = 3 \times (1+3) = 12.$$

All other triples satisfy the inequality trivially. Thus (X, d, s) is an extended b-metric space. Moreover, X is finite, so it is complete. The value $S = \sup\{s(x,y) : x, y \in X\} = 3$, and clearly $1 \leq S < \infty$. Therefore (X, d, s) is a CEBMS.

Since $S = 3$, we have

$$\frac{1}{2S+1} = \frac{1}{2\cdot 3+1} = \frac{1}{7} \approx 0.142857.$$

Define the altering distance function $\vartheta : [0,\infty) \to [0,\infty)$ by $\vartheta(t) = t$ for all $t \geq 0$. This function is continuous, strictly increasing, satisfies $\vartheta(t) = 0 \Leftrightarrow t = 0$,

Define the mapping T: $X \to X$ by

$$T(a) = b, \qquad T(b) = b, \qquad T(c) = b.$$

The Kannan-type condition: Choose $p = 0.1$. Clearly $p = 0.1 < \frac{1}{7} \approx 0.142857$, so the condition $p < \frac{1}{2S+1}$ is satisfied.

We now verify inequality (2.7) for all $x, y \in X$:

$$\vartheta(d(Tx,Ty)) \leq p\{\vartheta(d(x,y)) + \vartheta(d(x,Tx)) + \vartheta(d(y,Ty))\}.$$

Since $\vartheta(t) = t$, this becomes:

$$d(Tx,Ty) \leq p\{d(x,y) + d(x,Tx) + d(y,Ty)\}.$$

Case 1: $x = a$, $y = c$.

$$d(Ta,Tc) = d(b,b) = 0,$$
$$p\{d(a,c) + d(a,Ta) + d(c,Tc)\} = 0.1\times\{3+1+1\} = 0.1\times 5 = 0.5.$$

Thus $0 \leq 0.5$.

Case 2: $x = a$, $y = b$.

$$d(Ta,Tb) = d(b,b) = 0,$$
$$p\{d(a,b) + d(a,Ta) + d(b,Tb)\} = 0.1\times\{1+1+0\} = 0.1\times 2 = 0.2.$$

Thus $0 \leq 0.2$.

Case 3: $x = b$, $y = c$.

$$d(Tb,Tc) = d(b,b) = 0,$$
$$p\{d(b,c) + d(b,Tb) + d(c,Tc)\} = 0.1\times\{1+0+1\} = 0.1\times 2 = 0.2.$$

Thus $0 \leq 0.2$.

All remaining cases follow by symmetry or are trivial. Hence the Kannan-type condition holds for all $x, y \in X$ with $p = 0.1$.

Since $T(b) = b$, the point $b \in X$ is a fixed point of T. To show uniqueness, suppose $z \in X$ is another fixed point of T. Then $T(z) = z$. But $T(x) = b$ for all $x \in X$, so $z = b$. Hence b is the unique fixed point.

This example provides a concrete situation where the classical metric framework fails, while the extended b-metric structure allows the application of fixed point theory, thereby illustrating the true generalization achieved in this work.

Theorem 2.5. *Let (X, d, s) be a (CEBMS) with* $S = \sup\{s(x,y) : x, y \in X\}$, *where* $1 \leq S < \infty$. *Let* $\vartheta : [0,\infty) \to [0,\infty)$ *be an altering distance function as in Definition 1.3, and let* $T : X \to X$ *be a mapping such that for some constant* $p \in \left[0, \frac{1}{2S}\right)$,

$$\vartheta(d(Tx,Ty)) \leq p\{\vartheta(d(x,Tx)) + \vartheta(d(y,Ty))\}$$

for all $x, y \in X$. *Then T has a unique fixed point* $\omega \in X$, *and for every* $x_0 \in X$, *the Picard sequence* $\{T^n x_0\}$ *converges to* ω.

Proof. The result follows directly from Theorem 2.1. □

3. Application

Fixed point theory has gained increasing importance in engineering and scientific applications, particularly in the analysis of fractional-order chaotic systems [2], the modeling of heat transfer dynamics in complex fluid layers [20], and the development of proximal point results with broad applicability [19]. Motivated by these developments, we present the following application of our main results to Fredholm integral equations. This section the existence and novelty of answers to the FIE in the context of CEBMS. Consider the FIE

$$x(t) = f(t) + \gamma\int_{\kappa}^{\sigma} K(t,s,x(s))ds, \qquad t \in [\kappa,\sigma] \tag{3.1}$$

where $x \in C[\kappa,\sigma]$ and $f:[\kappa,\sigma] \to \mathbb{R}$ is continuous and $K:[\kappa,\sigma]\times[\kappa,\sigma]\times\mathbb{R} \to \mathbb{R}$ is continuous.

Let $X = C[\kappa,\sigma]$ and define the EBMS function $d: X\times X \to [0,\infty)$ by

$$d(x,y) = \sup_{t\in[\kappa,\sigma]} |x(t)-y(t)| \tag{3.2}$$

with the coefficient function $s: X\times X \to [1,\infty)$ given by

$$s(x,y) = 2 + \frac{1}{1+|x-y|} \quad \Rightarrow \quad S = \sup s(x,y) = 3.$$

Definition 3.1. Let (X, d, s) be a CEBMS and $T: X \to X$ be a mapping. The fixed point problem

$$x = Tx \tag{3.3}$$

is said to be Hyers-Ulam stable (HUS) if and only if for every $x \in X$ fulfilling the inequality

$$d(x,Tx) \le \epsilon \tag{3.4}$$

and the inequality $d(Tx,x) \le \epsilon$, where $\varepsilon > 0$, there exists a solution $x^* \in X$ and a constant $W > 0$ independent of x and x^* such that

$$d(x^*,x) \le W\varepsilon, \qquad d(x,x^*) \le W\varepsilon. \tag{3.5}$$

Theorem 3.1. *Let $X = C[\kappa,\sigma]$ be equipped with the metric*

$$d(x,y) = \sup_{t\in[\kappa,\sigma]} |x(t)-y(t)|.$$

Define the coefficient function $s: X\times X \to [1,\infty)$ by

$$s(x,y) = 2 + \frac{1}{1+d(x,y)}.$$

Then $S = \sup_{x,y\in X} s(x,y) = 3,$ and $1 \le S < \infty$.

Consider the Fredholm integral equation

$$x(t) = f(t) + \gamma\int_{\kappa}^{\sigma} K(t,s,x(s))ds, \qquad t \in [\kappa,\sigma],$$

and define the operator $T: X \to X$ by

$$(Tx)(t) = f(t) + \gamma\int_{\kappa}^{\sigma} K(t,s,x(s))ds.$$

Assume there exists $L > 0$ such that

$$|K(t,s,u)-K(t,s,v)|\le L|u-v|, \qquad \forall t,s\in[\kappa,\sigma],\, u,v\in\mathbb{R}.$$

Let $p=|\gamma|L(\sigma-\kappa)$. *If*

$$|\gamma|<\frac{1}{7L(\sigma-\kappa)},$$

then $p<\frac{1}{7}$ *and consequently* $Sp=3p<\frac{3}{7}<1$. *Then the following statements hold:*

- *(i) The operator T has a unique fixed point* $x \in X$*, which is the unique solution of the FIE.*
- (ii) *The fixed point* x^* *is Hyers-Ulam stable. More precisely, for any* $\varepsilon > 0$ *and any* $y \in X$ *satisfying* $d(y, Ty) \in \varepsilon$*, we have*

$$d(x^*,y)\le\frac{S}{1-Sp}\varepsilon=\frac{3}{1-3p}\varepsilon.$$

Proof. (i) For all $x,y\in X$:

$$\begin{aligned}
d(Tx,Ty)&=\sup_{t\in[\kappa,\sigma]}|(Tx)(t)-(Ty)(t)|\\
&=\sup_{t\in[\kappa,\sigma]}\left|\gamma\int_{\kappa}^{\sigma}[K(t,s,x(s))-K(t,s,y(s))]ds\right|\\
&\le|\gamma|\sup_{t\in[\kappa,\sigma]}\int_{\kappa}^{\sigma}|K(t,s,x(s))-K(t,s,y(s))|ds\\
&\le|\gamma|L\sup_{t\in[\kappa,\sigma]}\int_{\kappa}^{\sigma}|x(s)-y(s)|ds\\
&\le|\gamma|L(\sigma-\kappa)\sup_{t\in[\kappa,\sigma]}|x(s)-y(s)|\\
&=|\gamma|L(\sigma-\kappa)d(x,y).
\end{aligned}$$

Let $k=|\gamma|L(\sigma-\kappa)$, thus $k<\frac{1}{3}$. Utilizing the generalized Kannan fixed point theory:

$$\vartheta(d(Tx,Ty))\le p\{\vartheta(d(x,y))+\vartheta(d(x,Tx))+\vartheta(d(y,Ty))\}.$$

Take $\vartheta(t)=t$, then

$$\begin{aligned}
d(Tx,Ty)&\le kd(x,y)\\
&\le k\{d(x,y)+d(x,Tx)+d(y,Ty)\}.
\end{aligned}$$

Since $k<\frac{1}{3}$ and $S=\sup s(x,y)=3$, we have $k<\frac{1}{3}=\frac{1}{2S+1}$.

Consequently, all prerequisites of Theorem 2.1 are fulfilled. By Theorem 2.1 (Generalized Kannan Fixed Point Theorem in CEBMS, the operator T has a unique fixed point. This fixed point is the unique solution of the FIE. By Theorem 3.1, under the given assumptions, there exists a unique exact solution $x^*\in C[\kappa,\sigma]$ of the FIE, i.e., $x^* = Tx^*$.

Let $\varepsilon > 0$ and let $y \in X$ be an approximate solution satisfying $d(y, Ty) \le \varepsilon$. Using the extended triangle inequality:

$$\begin{aligned}
d(x,y)&=d(Tx^*,y)\\
&\le s(x,y)[d(Tx^*,Ty)+d(Ty,y)]\\
&\le S[d(Tx^*,Ty)+\epsilon]
\end{aligned}$$

$$\begin{aligned} d(Tx^*,Ty) &= \sup_{t\in[\kappa,\sigma]} |(Tx^*)(t)-(Ty)(t)| \\ &= \sup_{t\in[\kappa,\sigma]} \left|\gamma\int_\kappa^\sigma [K(t,s,x^*(s))-K(t,s,y(s))]ds\right| \\ &\le |\gamma| \sup_{t\in[\kappa,\sigma]} \int_\kappa^\sigma |x^*(s)-y(s)|ds \\ &\le |\gamma| L(\sigma-\kappa) \sup_{t\in[\kappa,\sigma]} \left|x^*(s)-y(s)\right| \\ &= |\gamma| L(\sigma-\kappa) d(x^*,y) \\ &= pd(x^*,y). \end{aligned}$$

Therefore

$$\begin{aligned} d(x^*,y) &\le s(x^*,y)[d(x^*,Ty)+d(Ty,y)] \\ &\le S[d(Tx^*,Ty)+\epsilon] \\ &\le S[pd(x^*,y)+\epsilon] \end{aligned}$$

Thus,

$$d(x^*,y) \le Spd(x^*,y)+S\epsilon.$$
$$d(x^*,y)-Spd(x^*,y) \le S\epsilon,$$
$$(1-Sp)d(x^*,y) \le S\epsilon.$$

Since $1 - Sp > 0$, we can divide:

$$d(x^*,y) \le \frac{S}{1-Sp}\epsilon.$$

By using Condition 3.5 in Definition 3.1.

$$W = \frac{S}{1-Sp} > 0.$$

Then $d(x^*,y) \le W\epsilon$, which proves that the fixed point x^* is Hyers-Ulam stable. ☐

Example 3.2. Consider the Fredholm integral equation

$$x(t) = f(t) + \lambda\int_0^1 K(t,s)x(s)ds, \qquad t\in[0,1],$$

where $f(t) = \frac{t}{6}, \lambda = \frac{1}{10}$, and the kernel $K(t, s) = ts$.

Let $X = C([0, 1])$ be equipped with the extended b-metric

$$d(x,y) = \sup_{t\in[0,1]} |x(t)-y(t)|^2$$

and the coefficient function $s(x, y) = 2$ for all $x, y \in X$. Then $S = \sup_{x,y\in X}\{s(x,y)\} = 2$, so $1 \le S < \infty$ and

$$\frac{1}{2S+1} = \frac{1}{5} = 0.2.$$

Define the operator $T: X \to X$ by

$$(Tx)(t) = \frac{t}{6} + \frac{t}{10}\int_0^1 sx(s)ds.$$

Let $c_x = \int_0^1 sx(s)ds$, then $(Tx)(t) = \frac{t}{6} + \frac{c_x t}{10}$.

Take $\vartheta(t) = t$ for all $t \geq 0$, which satisfies the properties of an altering distance function: it is continuous, strictly increasing, $\vartheta(t) = 0 \Leftrightarrow t = 0$, in 1.3. Choose $p = 0.15$. Since $p = 0.15 < 0.2$, the condition $p < \dfrac{1}{2S+1}$ is satisfied.

For any $x, y \in X$, we have

$$|c_x - c_y| = \left|\int_0^1 s(x(s) - y(s))ds\right| \leq \int_0^1 s\,|x(s) - y(s)|\,ds \leq \frac{1}{2}\sup_{s\in[0,1]} |x(s) - y(s)|,$$

because $\int_0^1 s\,ds = \dfrac{1}{2}$.

Since $d(x,y) = \sup|x - y|^2$, we have $\sup|x - y| = \sqrt{d(x,y)}$. Hence

$$(c_x - c_y)^2 \leq \frac{1}{4}d(x,y).$$

Now compute $d(Tx, Ty)$:

$$d(Tx,Ty) = \sup_{t\in[0,1]}\left|\frac{(c_x - c_y)t}{10}\right|^2 = \frac{(c_x - c_y)^2}{100}.$$

Since $(c_x - c_y)^2 \leq \dfrac{1}{4}d(x,y)$, we obtain

$$d(Tx,Ty) \leq \frac{1}{100}\cdot\frac{1}{4}d(x,y) = \frac{1}{400}d(x,y).$$

Now verify the Kannan-type condition 2.1:

$$d(Tx,\, Ty) \leq p\{d(x,y) + d(x,Tx) + d(y,Ty)\}.$$

Since $d(Tx,Ty) \leq \dfrac{1}{400}d(x,y)$ and the right-hand side is at least $\dfrac{1}{400}d(x,y)$ (because $pd(x,y) \geq \dfrac{1}{400}d(x,y)$ when $d(x,y) \neq 0$), the condition holds. For the worst case $x(t) = t$, $y(t) = 0$:

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{400} = 0.0025,\\ d(x,y) &= 1,\\ d(x,Tx) &= \sup_t\left|t - \frac{t}{6} - \frac{t}{20}\right|^2 = \left(\frac{43}{60}\right)^2 \approx 0.514,\\ d(y,Ty) &= \sup_t\left|\frac{t}{6}\right|^2 = \frac{1}{36} \approx 0.028. \end{aligned}$$

Then $p(1 + 0.514 + 0.028) = 0.15 \times 1.542 = 0.2313 \geq 0.0025$, so the condition 2.1 is satisfied.

The unique fixed point satisfies $x^*(t) = \dfrac{t}{6} + \dfrac{c^* t}{10}$, where $c^* = \int_0^1 s x^*(s)ds$. Substituting:

$$c^* = \int_0^1 s\left(\frac{s}{6} + \frac{c^* s}{10}\right)ds = \frac{1}{18} + \frac{c^*}{30}.$$

Solving gives $c^* = \dfrac{5}{87}$, and therefore

$$x^*(t) = \frac{t}{6} + \frac{t}{10}\cdot\frac{5}{87} = \frac{t}{6} + \frac{t}{174} = \frac{30t}{174} = \frac{5t}{29}.$$

Starting from $x_0(t) = 0$, the Picard iterates are $x_n(t) = \alpha_n t$, where $\alpha_{n+1} = \frac{1}{6} + \frac{\alpha_n}{30}$. The error estimate from Theorem 2.1 gives

$$d(T^{n+1}x_0, T^n x_0) \le q^n d(x_0, Tx_0), \quad q = \frac{2p}{1-p} = \frac{0.3}{0.85} \approx 0.353.$$

Since $d(x_0, Tx_0) = \frac{1}{36}$, the numerical results are:

n	α_n	$x_n(1)$	$\lvert x_n(1) - x^*(1) \rvert$
0	0.00000	0.00000	$\lvert$ 0.00000 – 0.17241 $\rvert$ = 0.17241
1	0.16667	0.16667	$\lvert$ 0.16667 - 0.17241 $\rvert$ = 0.00574
2	0.17222	0.17222	$\lvert$ 0.17222 - 0.17241 $\rvert$ = 0.00019
3	0.17241	0.17241	$\lvert$ 0.17241 - 0.17241 $\rvert$ = 0.00000
4	0.17241	0.17241	$\lvert$ 0.17241 - 0.17241 $\rvert$ = 0.00000

where $x^*(1) = \frac{5}{29} \approx 0.17241$. The sequence converges rapidly to the unique solution.

The Hyers-Ulam stability constant for this example is

$$W = \frac{S}{1 - Sp} = \frac{2}{1 - 2(0.15)} = \frac{2}{0.7} \approx 2.857,$$

which is well-defined since $Sp = 0.3 < 1$.

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