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Self-Similar sets with single-point intersections violating OSC

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Abstract

In the one-dimensional Euclidean space, it was shown that there exists a self-similar set with a single-point intersection that does not satisfy the open set condition (OSC). In the present paper, we prove the existence of a self-similar set with analogous properties in which all mappings are of the form $S_i(x) = q_i x + a_i$ with $q_i > 0$.

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Key words and Phrases: Self-similar sets, Single intersection property, Open Set Condition, Cantor discontinuum.

1. Introduction

Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a system of contractive similarities in the Euclidean space $\mathbb{R}^n$. According to Hutchinson's theorem, there exists a unique nonempty compact set K satisfying the invariant equation $K = \bigcup_{i=1}^{m} S_i(K)$, which is called the *attractor* or the *self-similar set* generated by the system $\mathcal{S}$. The system $\mathcal{S}$ satisfies the open set condition (OSC) if there exists a nonempty open set $\mathcal{O}$ such that $S_i(\mathcal{O}) \subset \mathcal{O}$ and $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \varnothing$ for all $i \neq j$. The open set condition was introduced by Moran in [5] in 1946 and became widely known after the work of Hutchinson [3] in 1981. In particular, the open set condition holds if the system $\mathcal{S}$ satisfies the strong separation condition, under which the first-level copies $S_i(K)$, i =1,2, ..., m, are pairwise disjoint. A minimal weakening of this condition requires that any two distinct copies of the attractor have at most one point in common. Such a condition yields systems with a single-point intersection, and the most minimal case occurs when, in the system $\mathcal{S}$, only two

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prescribed copies have a nonempty intersection consisting of exactly one point. In [7], an example of a totally disconnected perfect self-similar subset of the interval [0, 1] was constructed, in which the copies K_i and K_j are oppositely oriented and intersect at a single point, namely $S_i(1) = S_j(0) = \max K_i = \max K_j$.

In the present paper, we construct an example of a system on the interval [0, 1] that has the property of a unique single-point intersection, consists of mappings oriented in the same direction, and does not satisfy the open set condition.

Theorem 1.1. *There exists a system $\mathcal{S}$ of contractive similitudes in $\mathbb{R}$ for which the following conditions hold:*

1) All mappings of the system $\mathcal{S}$ are oriented in the same direction;
2) The system $\mathcal{S}$ does not satisfy the open set condition;
3) The copies of the attractor intersect at a unique point, which is the image of the fixed points of two mappings of the system $\mathcal{S}$.

2. Preliminaries

Definition 2.1. *Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a system of contractive similarities in $\mathbb{R}^n$. A nonempty compact set K is called a* self-similar *set or the* attractor *of the system $\mathcal{S}$ if $K = \bigcup_{i=1}^{m} S_i(K)$.*

Let $I = \{1, ..., m\}$ be the index set of the system $\mathcal{S}$. Denote by $I^* = \bigcup_{n=1}^{\infty} I^n$ the set of all finite words $\mathbf{i} = i_1 \cdots i_n$ over the alphabet I, which we call multiindices. The set $I^\infty = \{1,2,\ldots,m\}^{\mathbb{N}}$ is called the *index space*, and its elements are denoted by $\alpha = \alpha_1\alpha_2\cdots,\alpha_k \in I$. The mapping $\pi : I^\infty \to K$ is called the *index map*; it assigns to each element of I^∞ the corresponding point $x = \bigcap_{n=1}^{\infty} K_{\alpha_1 \ldots \alpha_n}$ of the attractor K. For a multiindex $\mathbf{j} = j_1 j_2 ..., j_n$, we use the notation $S_{\mathbf{j}} = S_{j_1} \circ \cdots \circ S_{j_n}$, and we denote the image of the attractor K under the mapping $S_{\mathbf{j}}$ by $K_{\mathbf{j}}$. For two multiindices $\mathbf{i}, \mathbf{j} \in I^*$, we write $\mathbf{i} \sqsubset \mathbf{j}$ if there exists a multiindex $\mathbf{k} \in I^*$ such that $\mathbf{j} = \mathbf{ik}$, that is, $\mathbf{i}$ is a prefix of the word $\mathbf{j}$. Given a vector of contraction coefficients $q = (q_1,\ldots,q_m)$ with $0 < q_i < 1$, a metric ρ_q is defined on the space of infinite sequences I^∞ by $\rho_q(\alpha, \beta) = q_{\alpha\wedge\beta}$ (see [2]), where $\alpha \wedge \beta$ denotes the longest common prefix of the sequences α and β. We denote the space I^∞ endowed with this metric by $I^\infty_{\rho_q}$. Let $s_q = \dim_H I^\infty_{\rho_q}$. Then, according to [2, Theorem 6.4.3], this quantity s_q is determined as the solution of the Moran equation $\sum_{i=1}^{m} q_i^{s_q} = 1$.

Definition 2.2. *We say that the system $\mathcal{S}$ satisfies the open set condition (OSC) if there exists a nonempty open set $\mathcal{O} \subset \mathbb{R}^n$ such that $S_i(\mathcal{O}) \subset \mathcal{O}$ for all $i = 1,\ldots,m$, and $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \varnothing$ whenever $i \neq j$.*

Let $\mathcal{F} = \{S_{\mathbf{i}}^{-1} S_{\mathbf{j}} \mid \mathbf{i}, \mathbf{j} \in I^*\}$. We say that the system $\mathcal{S}$ satisfies the weak separation property (WSP) if the identity map Id is an isolated point of the set $\mathcal{F}$ [8]. If the system does not have WSP, then it does not satisfy OSC, but the opposite is not true.

Definition 2.3. *The system $\mathcal{S}$ is said to satisfy the single intersection property (SIP) if $\#(K_i \cap K_j) \le 1$ for all $i \neq j$.*

General Position Theorem. We state the general position theorem proved in [1]. Note that in what follows, we will use a simplified version of this theorem, which is sufficient for solving the problems considered in the present work.

Theorem 2.4. *Let* (D,ρ_D), (L_1,ρ_1), *and* (L_2,ρ_2) *be metric spaces. Let* $\varphi_1(\tau,x): D\times L_1 \to \mathbb{R}^n$ *and* $\varphi_2(\tau,x): D\times L_2 \to \mathbb{R}^n$ *be continuous functions such that*

(1) there exist constants $C>0$ *and* $\alpha>0$ *such that*

$$\|\varphi_i(\tau,x)-\varphi_i(\tau,x')\| \le C\rho_i(x,x')^\alpha$$

for all $\tau\in D$, $i=1,2$, *and all* $x,x'\in L_i$;

(2) there exists a constant $M>0$ *such that for every* $x_i\in L_i$, $i=1,2$, *and all* $\tau_1,\tau_2\in D$, *the function*

$$\varPhi(\tau,x_1,x_2)=\varphi_1(\tau,x_1)-\varphi_2(\tau,x_2)$$

satisfies the inequality

$$\|\varPhi(\tau_2,x_1,x_2)-\varPhi(\tau_1,x_1,x_2)\| \ge M\rho_D(\tau_1,\tau_2).$$

Then, for the set $\varDelta := \{\tau\in D: \varphi_1(\tau,L_1)\cap\varphi_2(\tau,L_2)\ne\varnothing\}$, *the following estimate holds:*

$$\dim_H \varDelta \le \min\left\{\frac{\dim_H(L_1\times L_2)}{\alpha}, \dim_H D\right\}.$$

Lemma 2.5. *Let* $\mathcal{S}=\{S_1, ..., S_m\}$ *and* $\mathcal{S}'=\{S'_1, ..., S'_m\}$ *be systems of contractive mappings in a metric space* (X,ρ) *with attractors* K *and* K', *and with index parametrizations* $\pi: I^\infty\to K$ *and* $\pi': I^\infty\to K'$, *respectively. Assume that* $V\subset X$ *is a compact set such that* $S_i(V)\subset V$ *and* $S_i'(V)\subset V$ *for all* $i\in I$. *Then, for any* $\xi\in I^\infty$, *we have*

$$\rho\left(\pi(\xi),\pi'(\xi)\right)\le\frac{R}{1-\overline{r}},$$

where $R=\max\limits_{x\in V, i\in I}\rho\left(S_i'(x),S_i(x)\right)$, $\overline{r}=\max\limits_{i\in I}\{\operatorname{Lip}S_i,\operatorname{Lip}S_i'\}$.

3. Construction of a family of self-similar sets with a single-point intersection in $\mathbb{R}$

Let $p\in(0.003, 0.015)$. We introduce a family of contractive similarities in $\mathbb{R}$, depending on p, denoted by $\mathcal{S}_p=\{S_1, ..., S_6\}$, which are defined by the following equations:

$$S_1(x)=\frac{x}{56},\quad S_2(x)=\frac{x}{56}+\frac{1}{8}\quad S_3(x)=p\left(x-\frac{6}{7}\right)+\frac{1}{2},\quad S_4(x)=\frac{x}{56}+\frac{1}{2},$$
$$S_5(x)=\frac{x}{36}+\frac{5}{6},\quad S_6(x)=\frac{x}{56}+\frac{55}{56}.$$

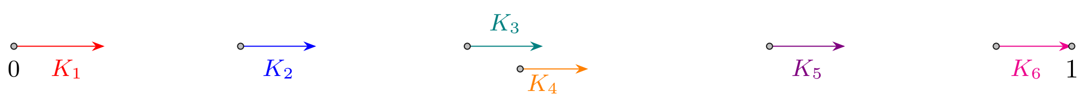

Figure 1. Schematic configuration of the first-level copy K_i in the attractor K_p.

Denote the attractor of the system $\mathcal{S}_p$ by K_p, and denote its first-level copies by $K_i=S_i(K_p)$, $i=1, ..., 6$. By the construction of the system, among the copies K_i, only K_3 and K_3 have a nonempty intersection.

Note that $z_1=0$ is the fixed point of the map S_1, and $z_5=\dfrac{6}{7}$ is the fixed point of S_5. Moreover, $z:=\dfrac{1}{2}=S_3(z_5)=S_4(z_1)$. We aim to show that for almost all $p\in(0.003, 0.015)$, one has $K_3\cap K_4=\{z\}$. In this case, $\mathcal{S}_p$ is a system with a single-point intersection.

Lemma 3.1. *For the system* $\mathcal{S}_p$, *the weak separation property does not hold for any* p.

Proof. Let $G_m(x)=S_4S_1^mS_2(x)$ and $G_n(x)=S_3S_5^nS_6(x)$. We show that there exist sequences $\{m_k\}$ and $\{n_k\}$ such that $G_{m_k}^{-1}\circ G_{n_k}$ converges to the identity map Id.

A direct computation yields

$$G_m(x)=\frac{x}{56^{m+2}}+\frac{1}{8\cdot 56^{m+1}}+\frac{1}{2},\qquad G_n(x)=\frac{px}{56\cdot 36^n}+\frac{p}{8\cdot 36^n}+\frac{1}{2}.$$

Since $\frac{\log 56}{\log 36}\notin\mathbb{Q}$, it follows from [4, Lemma 7] that for any $p>0$, there exist sequences $\{m_k\}$ and $\{n_k\}$ such that $\lim_{k\to\infty}\frac{36^{n_k}}{56^{m_k+1}}=p$. Consider the sequence

$$G_{m_k}^{-1}\circ G_{n_k}(x)=\frac{p56^{m_k+1}}{36^{n_k}}x+\frac{p56^{m_k+2}}{8\cdot 36^{n_k}}-\frac{56}{8}.$$

Then $G_{m_k}^{-1}\circ G_{n_k}(x)\to x$ as $k\to\infty$. Hence, Id is not an isolated point of the family $\mathcal{F}$. □

4. A system $\mathcal{S}_p$ with a single-point intersection

In the proof of the equality $K_3\cap K_4=\{z\}$, we rely on the general position theorem. Let

$$\Delta=\{p\in(0.003,0.015): K_3\cap K_4\neq\{z\}\}.$$

Clearly,

$$K_p=\{z_1\}\cup\bigcup_{m=0}^{\infty}S_1^m\left(K\setminus K_1\right)=\{z_5\}\cup\bigcup_{n=0}^{\infty}S_5^n\left(K\setminus K_5\right),$$

and hence

$$K_3=\{z\}\cup\bigcup_{n=0}^{\infty}S_3S_5^n\left(K\setminus K_5\right),\qquad K_4=\{z\}\cup\bigcup_{m=0}^{\infty}S_4S_1^m\left(K\setminus K_1\right).$$

Then $K_3\cap K_4=\{z\}$ if and only if for all $m,n\in\mathbb{N}\cup\{0\}$ and all $i\in I\setminus\{1\}$, $j\in I\setminus\{5\}$ one has $S_4S_1^m(K_i)\cap S_3S_5^n(K_j)=\varnothing$. Accordingly, we may write $\Delta=\{p\in(0.003,0.015): S_4S_1^m(K_i)\cap S_3S_5^n(K_j)\neq\varnothing,$ $i\in I\setminus\{1\}, j\in I\setminus\{5\}\}$. For $x\in K_p\setminus K_1$, $y\in K_p\setminus K_5$, define $\varphi_1(p,x)=S_4S_1^mS_i(x)$, $\varphi_2(p,y)=S_3S_5^nS_j(y)$. In this setting, we need to estimate the Hausdorff dimension of the set Δ and show that $\dim_H\Delta<1$.

First, we verify condition (1) of Theorem 2.4.

Let $I^\infty=\{1,2,3,4,5,6\}^{\mathbb{N}}$. Denote by $q_i=\mathrm{Lip}\ S_i$, $1\le i\le 6$ the Lipschitz constants of the mappings in the system $\mathcal{S}_p$. For the vector $(q_1, ..., q_6)$, we define a metric $\rho_q(\xi,\eta)$ on the space I^∞, and denote by $I^\infty_{\rho_q}$ the index space I^∞ endowed with this metric. Since $p<0.015$, we have $\dim_H I^\infty_{\rho_q}<0.46$.

Consider the index map $\pi_p: I^\infty_{\rho_q}\to K_p$.

Lemma 4.1. ([4, lemma 18]) *The map $\pi_p: I^\infty_{\rho_q}\to K_p$ is Lipschitz.*

Proof. Let $\xi,\eta\in I^\infty_{\rho_q}$. Assume that their longest common prefix is $\xi\wedge\eta=i_1i_2\cdots i_k$. Then, by the definition of the metric, $\rho_q(\xi,\eta)=q_{i_1i_2\cdots i_k}$. There exists a word $\mathbf{i}\in I^k$ such that $\mathbf{i}\sqsubset\xi$ and $\mathbf{i}\sqsubset\eta$. Consequently, the points $\pi_p(\xi)$ and $\pi_p(\eta)$ are contained in the set $S_\mathbf{i}(K_p)$. Since $\mathrm{Lip}(S_\mathbf{i})=q_{i_1i_2\cdots i_k}$, we obtain $\frac{|\pi_p(\xi)-\pi_p(\eta)|}{\rho_q(\xi,\eta)}<1.$ □

By Lemma 4.1, it follows that condition (1) is satisfied for the mappings $S_4S_1^mS_i\left(\pi_p(\xi)\right)$ and $S_3S_5^nS_j\left(\pi_p(\eta)\right)$, where $\xi,\eta\in I^\infty_{\rho_q}$.

We now verify condition (2) of Theorem 2.4.

It is clear that if $S_4S_1^m(K_i)\cap S_3S_5^n(K_j)\neq\varnothing$ for $i\in I\setminus\{1\}$ and $j\in I\setminus\{5\}$, then $S_4S_1^m([1/8,1])\cap S_3S_5^n([55/56,1])\neq\varnothing$. This condition is equivalent to

$$\frac{1}{56^{m+1}}\left[\frac{1}{8},1\right]\cap\frac{p}{36^n}\left[\frac{1}{8},\frac{1}{7}\right]\neq\varnothing.$$

Consequently, we obtain the inequality

$$\frac{7\cdot 36^n}{8\cdot 56^{m+1}}\le p\le\frac{8\cdot 36^n}{56^{m+1}}.$$

This inequality allows us to determine for which $m,n\in\mathbb{N}$ the copies $S_4S_1^m(K_p\setminus K_1)$ and $S_3S_5^n(K_p\setminus K_5)$ may intersect.

Lemma 4.2. *For any $\xi\in I^\infty$ and any $p_1,p_2\in(0.003,0.015)$, we have*

$$\left\|\pi_{p_1}(\xi)-\pi_{p_2}(\xi)\right\|\le|p_1-p_2|.$$

Proof. In the system, only the mapping S_3 depends on the parameter p. More precisely,

$$\left|S_{3,p_1}(x)-S_{3,p_2}(x)\right|\le\left|p_1\left(x-\frac{6}{7}\right)-p_2\left(x-\frac{6}{7}\right)\right|\le\frac{6}{7}|p_1-p_2|.$$

Consequently, $\max\limits_{x\in K_p,\, i\in I}\left|S_{i,p_1}(x)-S_{i,p_2}(x)\right|\le\frac{6}{7}|p_1-p_2|$. By Lemma 2.5, this implies that

$$\left\|\pi_{p_1}(\xi)-\pi_{p_2}(\xi)\right\|\le 0.89\,|p_1-p_2|.$$

□

Lemma 4.3. *Let $i\in I\setminus\{1\}$ and $j\in I\setminus\{5\}$. Define the mappings $\varphi_1(p,\xi)=S_4S_1^mS_i\left(\pi_p(\xi)\right)$, $\varphi_2(p,\eta)=S_3S_5^nS_j\left(\pi_p(\eta)\right)$, where $\xi,\eta\in I_{\rho_q}^\infty$. Then, for all $\xi,\eta\in I_{\rho_q}^\infty$ and for any $p_1,p_2\in(0.003,0.015)$, the following estimate holds:*

$$\left|\varphi_1(p_1,\xi)-\varphi_2(p_1,\eta)-\varphi_1(p_2,\xi)+\varphi_2(p_2,\eta)\right|\ge C|p_1-p_2|.$$

Proof. Let $i\in I\setminus\{1\}$, $j\in I\setminus\{5\}$, and let $\xi,\eta\in I^\infty$. Define $x_1=S_i\left(\pi_{p_1}(\xi)\right)$, $x_2=S_i'\left(\pi_{p_2}(\xi)\right)$, $y_1=S_j\left(\pi_{p_1}(\eta)\right)$, $y_2=S_j'\left(\pi_{p_2}(\eta)\right)$. Clearly, $|x_1-x_2|\le 0.89\,|p_1-p_2|$, $|y_1-y_2|\le 0.89\,|p_1-p_2|$.

Consider

$$\begin{aligned}&\left|S_4S_1^m(x_1)-S_{3,p_1}S_5^n(y_1)-S_4S_1^m(x_2)+S_{3,p_2}S_5^n(y_2)\right|=\\&\left|\frac{1}{56^{m+1}}(x_1-x_2)+\frac{1}{36^n}(p_2y_2-p_1y_1)+\frac{6}{7\cdot 36^n}(p_1-p_2)\right|=\\&\frac{1}{36^n}\left|\frac{36^n}{56^{m+1}}(x_1-x_2)+p_2(y_2-y_1)+y_1(p_2-p_1)+\frac{6}{7}(p_1-p_2)\right|\ge\\&\frac{1}{36^n}\left(\left|\left(\frac{6}{7}-y_1\right)(p_1-p_2)\right|-\left|p_2(y_2-y_1)\right|-\frac{36^n}{56^{m+1}}\left|x_1-x_2\right|\right)\ge\\&\frac{1}{36^n}\left(0.125|p_1-p_2|-0.03|p_1-p_2|\right)>\frac{0.09}{36^n}|p_1-p_2|.\end{aligned}$$

□

Lemma 4.4. *The system $\mathcal{S}_p$ has the single-point intersection property.*

Proof. Let $\Delta_{ij} = \{p \in (0.003, 0.015) : S_4 S_1^m(K_i) \cap S_3 S_5^n(K_j) \neq \varnothing\}$. Consider the mappings $\varphi_1(p,\xi) = S_4 S_1^m S_i(\pi_p(\xi))$, $\varphi_2(p,\eta) = S_3 S_5^n S_j(\pi_p(\eta))$, where $i \in I \setminus \{1\}$, $j \in I \setminus \{5\}$ and $\xi, \eta \in I_{\rho_q}^{\infty}$. Since all the assumptions of Theorem 2.4 are satisfied for the mappings φ_1 and φ_2, we obtain $\dim_H \Delta_{ij} \leq 2 \dim_H I_{\rho_q}^{\infty}$. Taking into account that

$$\Delta = \bigcup_{i,j} \Delta_{ij},$$

it follows that

$$\dim_H \Delta \leq \sup_{i,j} \dim_H \Delta_{ij} \leq 2 \dim_H I_{\rho_q}^{\infty} < 1.$$

Consequently, $S_4 S_1^m(K \setminus K_1) \cap S_3 S_5^n(K \setminus K_5) = \varnothing$ for almost all p. □

Theorem 4.5. *If the system $\mathcal{S}_p$ has the single-point intersection property, then* $\dim_H K_p = s$, *where s is determined as the unique solution of the equation*

$$4\left(\frac{1}{56}\right)^s + \left(\frac{1}{36}\right)^s + p^s = 1.$$

Proof. The proof of the theorem is analogous to the proof of [7, Theorem 9]. □

Conclusion

In this paper, we have constructed a one-dimensional system of contractive similitudes that provides an example of a self-similar set with a unique single-point intersection while failing to satisfy the open set condition. In contrast to previously known constructions, all mappings in our system are oriented in the same direction.

The proof of the single-point intersection property relies essentially on the general position theorem, which allows us to control the exceptional set of parameters and to show that, for almost all values of the parameter, the intersection of the corresponding copies of the attractor consists of exactly one point. Moreover, we establish that the weak separation property does not hold for the constructed system.

This example demonstrates that the single-point intersection phenomenon may occur even in the absence of the open set condition and without changing the orientation of the similarities.

References

[1] V. Aseev, K. Kamalutdinov, A.Tetenov, General Position Theorem and Its Applications. In: Devaney, R.L., Chan, K.C., Vinod Kumar, P. (eds) Topological Dynamics and Topological Data Analysis. IWCTA 2018. Springer Proceedings in Mathematics & Statistics, vol 350. Springer, Singapore.

[2] G. Edgar Measure, Topology, and Fractal Geometry, 2nd edition. New York: Springer-Verlag, (2008).

[3] J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. 30:5 (1981) 713–747.

[4] Kamalutdinov K. and Tetenov A.Twofold Cantor sets in ℝ, Siberian Electr. Math. Rep. 15 (2018), 801–814.

[5] P.A.P. Moran, Additive functions of intervals and Hausdorff measure, Proc. Cambridge Philos. Soc., 42 (1946), 15–23.

[6] A. Tetenov, K. Kamalutdinov, D. Vaulin Self-similar Jordan arcs which do not satisfy OSC, preprint: arXiv1512.00290, (2015).

[7] A. Tetenov, K. Kamalutdinov, On one-point intersection property for self-similar fractals, Nonlinearity. 33 (2020), 408–415.

[8] M. P. W. Zerner, Weak separation properties for self-similar sets, Proc. Amer. Math. Soc. 124,11 (1996), 3529–3539.