



M-systems of semirings applied to the bi-semirings via prime k-ideals

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Abstract

This paper aims to investigate certain notable classes of k-ideals of bi-semirings such as 0-prime, 1-prime, and 2-prime. Twenty sufficient and necessary conditions are established for unique type k-ideals to be type prime k-ideals in bi-semiring. A $T^1 - \text{KId}$ ($T^2 - \text{KId}$) in a bi-semiring was shown to correspond to a prime $T^1 - \text{KId}$ (prime $T^2 - \text{KId}$) in bi-semiring whenever it is either 0-prime 1-prime or 2-prime. There are numerous necessary and sufficient conditions for a $T^1 - \text{KId}$ ($T^2 - \text{KId}$) to be a $T^1 - \text{PKId}$ ($T^2 - \text{PKId}$). We show that when its complement is a τ_2^2 -system (τ_2^1), $T^1 - \text{KId}$ ($T^2 - \text{KId}$) is $T^1 - \text{PKId}$ ($T^2 - \text{PKId}$). There exists a $T^1 - \text{PKId}$ ($T^2 - \text{PKId}$) \mathfrak{P} such that $\mathfrak{P} \cap \mathfrak{Q} = \phi$ and $A \subseteq \mathfrak{P}$ for every $T^1 - \text{KId}$ ($T^2 - \text{KId}$) A in, where $A \cap \mathfrak{Q} = \phi$ where \mathfrak{Q} is τ_2^2 -system (τ_2^3 -system).

Key words and phrases. b-semirings prime ideals k-ideals m-systems

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1. Introduction

Semirings were first introduced in earlier research on the topic of ideals of rings. In terms of structure, semirings fall between rings and semigroups. Since a semiring is a generalization of a ring, it can be found in rings but not in semigroups. The study of rings demonstrates that, in contrast to the multiplicative structure of a semiring, the multiplicative structure of a ring is independent of its

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additive structure. The introduction of ideals for algebraic numbers and their subsequent application to associative rings were initially proposed by Dedekind. Ideals are important in the study of ring structures, and semirings are no exception. But compared to rings, semirings have very distinct principles. Many outcomes in rings have no equivalent in semirings, as evidenced by the absence of subtraction in semirings. An effort to make up for this lack of remove is the concept of k -ideal. Numerous investigations have been conducted since the advent of k -ideals [1, 2]. By extending results from rings and semirings and utilizing results from those earlier papers.

Dauns [3] studied one-sided prime ideals and prime right ideals of a ring. A number of conclusions on prime right ideals in a weakly regular ring were presented by Hansen [4]. There are several approaches to characterize prime ideals in semirings [5]. Sen et al. [6] attained this by using k -ideals to characterize a semiring. Based on semirings, Nandakumar [7] developed three different types of prime ideals. A system $(S, +, 0, \cdot, 1)$ is a (commutative) semiring if $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a commutative monoid, $0 \cdot x = 0 = x \cdot 0$ for all $x \in S$, and “ \cdot ” distributes over “ $+$ ”. k -ideals are ideals in semirings that have the very clear characteristics of ideals of rings: if $x + y \in I$ and $x \in I$, then $y \in I$. These ideals play an important role in semirings when additive inverses are present. The class of k -ideals is a more constrained class of ideals in semirings. These ideals have the characteristic that a subset of R is a k -ideal if and only if it is a ring ideal if the semiring R is a ring. The class of h -ideals is another class of ideals. In this study, we present a bi-semiring theory for T^1 -PKIDs that coincides for three different PIDs. We examine the relationships between the concepts associated with various PIDs and T^1 -PKIDs. A few of their features are analyzed, emphasizing the differences between various PIDs and their T^1 -PKID equivalents. Sen et al. [6] discussed the categorization of semirings into bi-semiring. We summarize numerous key concepts from the literature that are applied to semirings and bi-semirings in Section 2. Section 3 discusses the classification of Type-1 prime k -ideals and m -systems of bi-semirings. Section 4 describes the classification of Type-2 prime k -ideals and m -systems of bi-semirings.

2. Basic concepts

Definition 2.1. *The algebraic system $(\mathfrak{Z}, \alpha_1, \alpha_2, \alpha_3)$ is represent a bi-semiring if (\mathfrak{Z}, α_1) , (\mathfrak{Z}, α_2) and (\mathfrak{Z}, α_3) are semigroups connected by $\# \alpha_2 (b \alpha_1 \partial) = (\# \alpha_2 b) \alpha_1 (\# \alpha_2 \partial)$, $(\# \alpha_1 b) \alpha_2 \partial = (\# \alpha_2 \partial) \alpha_1 (b \alpha_2 \partial)$, $(b \alpha_2 \partial) = (\# \alpha_3 b) \alpha_2 (\# \alpha_3 \partial)$, $(\# \alpha_3 \partial) (\# \alpha_2 b) \alpha_3 \partial = (\# \alpha_3 \partial) \alpha_2 (b \alpha_3 \partial)$, $\forall \#, b, \partial \in \mathfrak{Z}$. That is $(\mathfrak{Z}, \alpha_1, \alpha_2, \alpha_3)$ is a bi-semiring if $(\mathfrak{Z}, \alpha_1, \alpha_2)$ is a T^1 -semiring and $(\mathfrak{Z}, \alpha_2, \alpha_3)$ is a T^2 -semiring.*

Definition 2.2. *Let ∇ and k be the subsets of $(\mathfrak{Z}, \alpha_1, \alpha_2, \alpha_3)$. Then their products $\nabla \alpha_1 k = \{\wp \alpha_1 \partial \mid \wp \in \nabla \text{ and } \partial \in k\}$, $\nabla \alpha_2 k = \{\wp \alpha_2 \partial \mid \wp \in \nabla \text{ and } \partial \in k\}$ and $\nabla \alpha_3 k = \{\wp \alpha_3 \partial \mid \wp \in \nabla \text{ and } \partial \in k\}$.*

Definition 2.3. *The subset ∇ in \mathfrak{Z} is represent a T^1 -RId (T^1 -LId) in \mathfrak{Z} if $\partial_1 \alpha_1 \partial_2 \in \nabla$ and $\partial_1 \alpha_2 \varsigma \in \nabla$ ($\varsigma \alpha_2 \partial_1 \in \nabla$) for each $\partial_1, \partial_2 \in \nabla$ and $\varsigma \in \mathfrak{Z}$. The subset ∇ in \mathfrak{Z} is represent a T^1 -Id in \mathfrak{Z} if it is both T^1 -RId and T^1 -LId in \mathfrak{Z} .*

Definition 2.4. *The subset ∇ in \mathfrak{Z} is represent a T^2 -RId (T^2 -LId) in \mathfrak{Z} if $\partial_1 \alpha_2 \partial_2 \in \nabla$ and $\partial_1 \alpha_3 \varsigma \in \nabla$ ($\varsigma \alpha_3 \partial_1 \in \nabla$) for each $\partial_1, \partial_2 \in \nabla$ and $\varsigma \in \mathfrak{Z}$. The subset ∇ in \mathfrak{Z} is represent a T^2 -Id in \mathfrak{Z} if it is both T^2 -RId and T^2 -LId in \mathfrak{Z} .*

Definition 2.5. *The subset ∇ in \mathfrak{Z} is represent a RId (LId) in \mathfrak{Z} if it is both T^1 -RId (T^1 -LId) and T^2 -RId (T^2 -LId) in \mathfrak{Z} . The subset ∇ in \mathfrak{Z} is represent a Id in \mathfrak{Z} if it is both T^1 -Id and T^2 -Id in \mathfrak{Z} .*

Lemma 2.6. *For any subsets \mathfrak{X} and \mathfrak{Y} of \mathfrak{Z} , which are closed under the operation “ α ”. Then $\mathfrak{X} \alpha \mathfrak{Y} \subseteq \mathfrak{X} \alpha \mathfrak{Y} \subseteq \mathfrak{Y} \subseteq \mathfrak{X} \alpha \mathfrak{Y} \subseteq \mathfrak{X} \alpha \mathfrak{Y}$*

Definition 2.7. *An ideal ∇ in a semiring S is represent a k -ideal if $\wp \in \nabla$, $\varsigma \in S$ and $\wp + \varsigma \in \nabla$ imply $\varsigma \in \nabla$.*

Definition 2.8. *The k -closure operation on semiring S is $C_k(\nabla) = \{\wp \in S \mid \wp + \varsigma \in \nabla \text{ for some } \varsigma \in \nabla\}$.*

3. Classification of Type-1 prime k-ideals and m-systems of bi-semirings

Definition 3.1. The subset \mathfrak{A} of \mathfrak{S} is represent a

- (i) $T^1 - RId$ ($T^1 - LId$) if $\varsigma \alpha_1 \eta \in \mathfrak{A}$ and $\varsigma \alpha_2 s \in \mathfrak{A}$ ($s \alpha_2 \varsigma \in \mathfrak{A}$) $\forall \varsigma, \eta \in \mathfrak{A}$ and $s \in \mathfrak{S}$.
- (ii) $T^2 - RId$ ($T^2 - LId$) if $\varsigma \alpha_2 \eta \in \mathfrak{A}$ and $\varsigma \alpha_3 s \in \mathfrak{A}$ ($s \alpha_3 \varsigma \in \mathfrak{A}$) $\forall \varsigma, \eta \in \mathfrak{A}$ and $s \in \mathfrak{S}$.

Lemma 3.2. (i) The $T^1 - RId$ generated by “ ς ” is $\langle \varsigma \rangle_{1r} = \{n \alpha_1 \varsigma \mid n \in \mathbb{Z}^+\} \cup \{\varsigma \alpha_2 \mathfrak{S}\}$.

- (ii) The $T^1 - LId$ generated by “ ς ” is $\langle \varsigma \rangle_{1l} = \{n \alpha_1 \varsigma \mid n \in \mathbb{Z}^+\} \cup \{\mathfrak{S} \alpha_2 \varsigma\}$.
- (iii) The $T^2 - RId$ generated by “ ς ” is $\langle \varsigma \rangle_{2r} = \{n \alpha_2 \varsigma \mid n \in \mathbb{Z}^+\} \cup \{\varsigma \alpha_3 \mathfrak{S}\}$.
- (iv) The $T^2 - LId$ generated by “ ς ” is $\langle \varsigma \rangle_{2l} = \{n \alpha_2 \varsigma \mid n \in \mathbb{Z}^+\} \cup \{\mathfrak{S} \alpha_3 \varsigma\}$.
- (v) If ∇ is a $T^1 - Id$ in \mathfrak{S} , then $\overline{\nabla}$ is a $T^1 - KId$ in \mathfrak{S} .
- (vi) If ∇ is a $T^1 - Id$ in \mathfrak{S} , then $\overline{\nabla} = \nabla$ if and only if ∇ is a $T^1 - KId$.

Definition 3.3. (i) The $T^1 - KId$ \mathfrak{A} is represent a $T^1 - PKId$ if $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$ for the $T^1 - KIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S}

- (ii) The $T^1 - KId$ \mathfrak{A} is represent a $0 - T^1 - PKId$ if $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$ for the $T^1 - Ids$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S}
- (iii) The $T^1 - KId$ \mathfrak{A} is represent a $1 - T^1 - PKId$ if $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$ for the $T^1 - KIds$ \mathfrak{A} and $T^1 - Id$ \mathfrak{J} in \mathfrak{S} .

Theorem 3.4. If \mathfrak{P} is a $T^1 - KId$ in \mathfrak{S} , then following conditions are equivalent.

- (i) \mathfrak{P} is $T^1 - PKId$.
- (ii) $\varsigma \alpha_2 \mathfrak{S} \alpha_2 \gamma \subseteq \mathfrak{P} \Leftrightarrow a \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (iii) (a)_{tri} α_2 (b)_{tri} $\subseteq \mathfrak{P} \Leftrightarrow a \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (iv) If the $T^1 - TRIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (v) (a) α_2 (b) $\subseteq \mathfrak{P} \Leftrightarrow a \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (vi) If the $T^1 - Ids$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (vii) If the $T^1 - KIds$ \mathfrak{A} and Ids \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (viii) If the $T^1 - Ids$ \mathfrak{A} and $KIds$ \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (ix) If the $T^1 - RIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (x) If the $T^1 - RKIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xi) If the $T^1 - RKIds$ \mathfrak{A} and $T^1 - RIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xii) If the $T^1 - RIds$ \mathfrak{A} and $T^1 - RKIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xiii) If the $T^1 - LIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xiv) If the $T^1 - LKIds$ \mathfrak{A} and \mathfrak{J} in \mathfrak{S} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xv) If the $T^1 - LKIds$ \mathfrak{A} and $T^1 - LIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xvi) If the $T^1 - LIds$ \mathfrak{A} and $T^1 - LKIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xvii) If the $T^1 - RIds$ \mathfrak{A} and $T^1 - LIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xviii) If the $T^1 - RKIds$ \mathfrak{A} and $T^1 - LKIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xix) If the $T^1 - RIds$ \mathfrak{A} and $T^1 - LKIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xx) If the $T^1 - RKIds$ \mathfrak{A} and $T^1 - LIds$ \mathfrak{J} , then $\mathfrak{A} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.

Proof. To prove the equivalence,

- (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)
- (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii)
- (vii) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (viii)
- (viii) \Leftrightarrow (xii) \Leftrightarrow (x) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (ix)
- (ix) \Leftrightarrow (x)(x) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (viii) \Leftrightarrow (vii) \Leftrightarrow (xi)
- (xi) \Leftrightarrow (x) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (viii) \Leftrightarrow (xii)
- (xii) \Leftrightarrow (x) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (xiii)(xiii) \Leftrightarrow (xiv)
- (xiv) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (xv)

- (xv) \hookrightarrow (xiv) \hookrightarrow (i) \hookrightarrow (ii) \hookrightarrow (v) \hookrightarrow (vi) \hookrightarrow (viii) \hookrightarrow (xvi)
- (xvi) \hookrightarrow (xiv) \hookrightarrow (i) \hookrightarrow (ii) \hookrightarrow (v) \hookrightarrow (vi) \hookrightarrow (xvii)
- (xvii) \hookrightarrow (xx) \hookrightarrow (xviii)(xviii) \hookrightarrow (i) \hookrightarrow (vi) \hookrightarrow (xix)
- (xix) \hookrightarrow (i) \hookrightarrow (vi) \hookrightarrow (xx). (i) \hookrightarrow (ii) \hookrightarrow (v) \hookrightarrow (vi) \hookrightarrow (vii) \hookrightarrow (i)
- (vi) \hookrightarrow (viii) \hookrightarrow (xii) \hookrightarrow (x) \hookrightarrow (i)
- (vi) \hookrightarrow (ix) \hookrightarrow (x)(vi) \hookrightarrow (vii) \hookrightarrow (xi) \hookrightarrow (x)
- (vi) \hookrightarrow (viii) \hookrightarrow (xvi) \hookrightarrow (xiv) \hookrightarrow (i)
- (vi) \hookrightarrow (xiii) \hookrightarrow (xiv)(vi) \hookrightarrow (vii) \hookrightarrow (xv) \hookrightarrow (xiv)
- (xvi) \hookrightarrow (xiv) \hookrightarrow (i) \hookrightarrow (vi) \hookrightarrow (xvii)
- (xvii) \hookrightarrow (xx) \hookrightarrow (xviii)(xviii) \hookrightarrow (i) \hookrightarrow (vi) \hookrightarrow (xix)
- (xix) \hookrightarrow (i) \hookrightarrow (vi) \hookrightarrow (xx).

(i) \Rightarrow (ii) : Let \mathfrak{P} be a PWKId. If $\epsilon \alpha_2 \mathfrak{Z} \alpha_2 \gamma \subseteq \mathfrak{P}$, then $\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z} \alpha_2 \mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z} \subseteq \mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z} \subseteq \mathfrak{Z} \alpha_2 \mathfrak{P} \alpha_2 \mathfrak{Z} \subseteq \mathfrak{P}$.

By Lemma 2.6 $\overline{\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z} \alpha_2 \mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z}} \subseteq \mathfrak{P}$. Hence, $\overline{\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z}} \subseteq \mathfrak{P}$ or $\overline{\mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z}} \subseteq \mathfrak{P}$. Assume $\overline{\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z}} \subseteq \mathfrak{P}$, then $\overline{(\epsilon) \alpha_2 (\epsilon) \alpha_2 (\epsilon)} \subseteq \overline{\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z}} \hookrightarrow \overline{(\epsilon)} \subseteq \mathfrak{P}$. Hence, $\epsilon \in \mathfrak{P}$. Similarly $\overline{\mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z}} \subseteq \mathfrak{P} \hookrightarrow \gamma \in \mathfrak{P}$.

(ii) \Rightarrow (v) : If $(\epsilon) \alpha_2 (\gamma) \subseteq \mathfrak{P}$, then $\epsilon \alpha_2 \mathfrak{Z} \alpha_2 \gamma \subseteq (\epsilon) \alpha_2 (\gamma) \subseteq \mathfrak{P}$. By (ii) $\epsilon \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.

(v) \Rightarrow (vi) : Let \mathfrak{R} and \mathfrak{J} be the T^1 -Ids in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$. If $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ $(\epsilon) \alpha_2 (\gamma) \subseteq \mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P} \hookrightarrow \gamma \in \mathfrak{P}$. Hence, $\mathfrak{J} \subseteq \mathfrak{P}$.

(vi) \Rightarrow (vii)(vii) \Rightarrow (i)(vi) \Rightarrow (viii)(xii) \Rightarrow (x) are direct.

(viii) \Rightarrow (xii) : If the T^1 -RId \mathfrak{R} and T^1 -RKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon) \alpha_2 (\gamma) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J}) \subseteq \mathfrak{P}$. By Lemma 3.2 and Lemma 2.6 $(\epsilon) \alpha_2 (\gamma) \subseteq (\epsilon) \alpha_2 (\gamma) \subseteq \mathfrak{P}$. Then by (viii) $\gamma \in \mathfrak{P} \forall \gamma \in \mathfrak{J}$. Hence, $\mathfrak{J} \subseteq \mathfrak{P}$.

(vi) \Rightarrow (ix) : Let \mathfrak{R} and \mathfrak{J} be the T^1 -RId of \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$. If $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\delta \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\chi \in \mathfrak{R}$ and by Lemma 3.2 $(\chi) \alpha_2 (\delta) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J}) \subseteq \mathfrak{P}$. Then by (vi) $\chi \in \mathfrak{P}$. Hence, $\mathfrak{R} \subseteq \mathfrak{P}$.

(ix) \Rightarrow (x) and (xi) \Rightarrow (x) are direct.

(x) \Rightarrow (i) : If the T^1 -RKIds \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon)_r \alpha_2 (\gamma)_r \subseteq \mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$.

By Lemma 2.6 $\overline{(\epsilon)_r \alpha_2 (\gamma)_r} \subseteq \overline{(\gamma)_r \alpha_2 (\gamma)_r} \subseteq \mathfrak{P}$. Hence, by (x) $\mathfrak{J} \subseteq \mathfrak{P}$.

(vii) \Rightarrow (xi) : If the T^1 -RKId \mathfrak{R} and T^1 -RId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\omega \in \mathfrak{R} \setminus \mathfrak{P}$ and $\nu \in \mathfrak{J} \setminus \mathfrak{P}$. Then $(\omega) \alpha_2 (\nu) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J}) \subseteq \mathfrak{P}$ and by Lemma 2.6 $(\omega) \alpha_2 (\nu) \subseteq (\omega) \alpha_2 (\nu) \subseteq \mathfrak{P}$, which contradicts (vii). Hence, (xi) holds.

(viii) \Rightarrow (xvi) : If the T^1 -LId \mathfrak{R} and T^1 -LKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\gamma \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\epsilon \in \mathfrak{R}$ and by Lemma 3.2 $(\epsilon) \alpha_2 (\gamma) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \subseteq \mathfrak{P}$. By Lemma 2.6 $(\epsilon) \alpha_2 (\gamma) \subseteq (\epsilon) \alpha_2 (\gamma) \subseteq \mathfrak{P}$. Hence, by (viii) $\mathfrak{R} \subseteq \mathfrak{P}$.

(xvi) \Rightarrow (xiv)(xiv) \Rightarrow (i)(xiii) \Rightarrow (xiv) are direct.

(vi) \Rightarrow (xiii) : If \mathfrak{R} and \mathfrak{J} are the T^1 -LIds of \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\chi \in \mathfrak{R} \setminus \mathfrak{P}$ and $\delta \in \mathfrak{J} \setminus \mathfrak{P}$. By Lemma 3.2 $(\chi) \alpha_2 (\delta) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \subseteq \mathfrak{P}$, which contradicts (vi). Hence, (xiii) holds.

(vii) \Rightarrow (xv) : If the T^1 -LKId \mathfrak{R} and T^1 -LId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\nu \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\omega \in \mathfrak{R}$ and by Lemma 3.2 $(\omega) \alpha_2 (\nu) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \subseteq \mathfrak{P}$. By Lemma 2.6 $(\omega) \alpha_2 (\nu) \subseteq (\omega) \alpha_2 (\nu) \subseteq \mathfrak{P}$. Hence, by (vii) $\mathfrak{R} \subseteq \mathfrak{P}$.

(xv) \Rightarrow (xiv)(xvii) \Rightarrow (xx)(xx) \Rightarrow (xviii) and (xviii) \Rightarrow (i) are direct.

(vi) \Rightarrow (xvii) : If the T^1 -RId \mathfrak{R} and T^1 -LId \mathfrak{I} in \mathfrak{Z} , then $\mathfrak{R} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon) \alpha_2 (\gamma) \subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \subseteq \mathfrak{P}$.

$$\begin{aligned}
 (\epsilon) \alpha_2 (\gamma) &= ((n \alpha_2 \epsilon) \cup (\epsilon \alpha_2 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_2 \epsilon) \cup (\mathfrak{Z} \alpha_2 \epsilon \alpha_2 \mathfrak{Z})) \alpha_2 ((m \alpha_2 \gamma) \cup (\gamma \alpha_2 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_2 \gamma) \cup (\mathfrak{Z} \alpha_2 \gamma \alpha_2 \mathfrak{Z})) \\
 &\subseteq (\mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{R} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{Z}) \\
 &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_2 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{P}) \cup (\mathfrak{Z} \alpha_2 \mathfrak{P} \alpha_2 \mathfrak{Z}) \\
 &\subseteq \mathfrak{P}.
 \end{aligned}$$

Hence, by (vi) $I \subseteq \mathfrak{P}$.

(vii) \Rightarrow (xix) : If the T^1 -RId \mathfrak{K} and T^1 -LKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{K} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\vartheta \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\varrho \in \mathfrak{K}$ and by Lemma 3.2 $(\varrho) \alpha_2 (\vartheta) \subseteq (\mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \subseteq \mathfrak{P}$. Now

$$\begin{aligned} (\varepsilon) \alpha_2 (\gamma) &= ((n \alpha_2 \varepsilon) \cup (\varepsilon \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \varepsilon) \cup (\mathfrak{J} \alpha_2 \varepsilon \alpha_2 \mathfrak{J})) \alpha_2 ((m \alpha_2 \gamma) \cup (\gamma \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \gamma) \cup (\mathfrak{J} \alpha_2 \gamma \alpha_2 \mathfrak{J})) \\ &\subseteq (\mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \\ &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{P}) \cup (\mathfrak{J} \alpha_2 \mathfrak{P} \alpha_2 \mathfrak{J}) \\ &\subseteq \mathfrak{P}. \end{aligned}$$

Now

$$\begin{aligned} (\varrho) \alpha_2 (\vartheta) &= ((n \alpha_2 \varrho) \cup (\varrho \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \varrho) \cup (\mathfrak{J} \alpha_2 \varrho \alpha_2 \mathfrak{J})) \alpha_2 ((m \alpha_2 \vartheta) \cup (\vartheta \alpha_2 \mathfrak{J}) \cup (\mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \varrho \alpha_2 \mathfrak{Z})) \\ &\subseteq (\mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \\ &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{P}) \cup (\mathfrak{J} \alpha_2 \mathfrak{P} \alpha_2 \mathfrak{J}) \\ &\subseteq \mathfrak{P}. \end{aligned}$$

Hence, by (vi) $\mathfrak{K} \subseteq \mathfrak{P}$.

(xix) \Rightarrow (i): is direct.

(vii) \Rightarrow (xx) : If the T^1 -RKId \mathfrak{K} and T^1 -LId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{K} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{K} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\varepsilon \in \mathfrak{K} \setminus \mathfrak{P}$ and $\gamma \in \mathfrak{J} \setminus \mathfrak{P}$. By Lemma 3.2 $(\varepsilon) \alpha_2 (\gamma) \subseteq (\mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J}) \cup (\mathfrak{J} \alpha_2 \mathfrak{K} \alpha_2 \mathfrak{J} \alpha_2 \mathfrak{J}) \subseteq \mathfrak{P}$, which contradicts (vi). Hence, (xx) holds.

Definition 3.5. The subset \mathfrak{D} of \mathfrak{Z} is represent

- (i) τ_0^2 -system if $\mathfrak{s}, \eta \in \mathfrak{D}$ and exists $s \in \mathfrak{Z} \hookrightarrow \zeta \alpha_2 s \alpha_2 \eta \in \mathfrak{D}$.
- (ii) τ_1^2 -system if $\mathfrak{s}, \eta \in \mathfrak{D}$ and exists $s \in (\underline{\zeta}), \eta \alpha_2 (\underline{\eta}) \hookrightarrow \zeta \alpha_2 \eta \in \mathfrak{D}$.
- (iii) τ_2^2 -system if $\mathfrak{s}, \eta \in \mathfrak{D}$ and exists $s \in (\zeta), \eta \alpha_2 (\eta) \hookrightarrow \zeta \alpha_2 \eta \in \mathfrak{D}$.

Lemma 3.6. The T^1 -KId \mathfrak{P} is a T^1 -PKId if and only if \mathfrak{P}^c is a $\tau_0^2(\tau_1^2\tau_2^2)$ -system.

Proof. Let \mathfrak{P} be an T^1 -PKId in \mathfrak{Z} and $\varepsilon \alpha_2 \gamma \in \mathfrak{P}^c$. Then by Theorem 3.4 $\varepsilon \alpha_2 \mathfrak{J} \alpha_2 \gamma \notin \mathfrak{P}$. Then there exists $\chi \in \mathfrak{Z}$, then $\varepsilon \alpha_2 \chi \alpha_2 \gamma \in \mathfrak{P}^c$. Hence, \mathfrak{P}^c is a τ_0^2 -system. For $\varepsilon \alpha_2 \gamma \in \mathfrak{P}^c$ by Theorem 3.4 (vii) and (x) $(\varepsilon) \alpha_2 (\gamma) \notin \mathfrak{P}$ and $(\varepsilon) \alpha_2 (\underline{\gamma}) \notin \mathfrak{P}$. Then \mathfrak{P}^c is a τ_1^2 -system and τ_2^2 -system respectively.

Conversely if there exist T^1 -KIds \mathfrak{K} and \mathfrak{J} , then $\mathfrak{K} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{K} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\varepsilon \in \mathfrak{K} \setminus \mathfrak{P}$ and $\gamma \in \mathfrak{J} \setminus \mathfrak{P}$. Hence, $\varepsilon \alpha_1 \gamma \in \mathfrak{P}^c \hookrightarrow \varepsilon_1 \alpha_2 \gamma_1 \in \mathfrak{P}^c$ for some $\varepsilon_1 \in (\varepsilon)$ and $\gamma_1 \in (\gamma)$, which contradicts $\mathfrak{K} \alpha_2 \mathfrak{J} \subseteq \mathfrak{P}$.

Remark 3.7. The Example 3.8 asserts that there are T^1 -KIds, which are not T^1 -PKIds.

Example 3.8. Consider the bi-semiring $\mathfrak{Z} = \{\natural, \hbar, \Delta, \wp, \triangleleft, \underline{\vee}\}$.

α_1	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
\natural	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
\hbar	\hbar	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
Δ	Δ	Δ	Δ	$\underline{\vee}$	\triangleright	$\underline{\vee}$
\wp	\wp	\wp	$\underline{\vee}$	\wp	\triangleright	$\underline{\vee}$
\triangleright	\triangleright	\triangleright	\triangleright	\triangleright	\triangleright	\triangleleft
$\underline{\vee}$	$\underline{\vee}$	$\underline{\vee}$	$\underline{\vee}$	$\underline{\vee}$	\triangleright	$\underline{\vee}$

α_2	\natural	\hbar	Δ	\wp	\triangleleft	$\underline{\vee}$
\natural	\natural	\natural	\natural	\natural	\natural	\natural
\hbar	\natural	\hbar	Δ	\hbar	Δ	Δ
Δ	\natural	\hbar	Δ	\hbar	Δ	Δ
\wp	\natural	\wp	\triangleright	\wp	\triangleright	\triangleright
\triangleright	\natural	\wp	\triangleright	\wp	\triangleright	\triangleright
$\underline{\vee}$	\natural	\wp	\triangleright	\wp	\triangleright	\triangleright

α_3	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
\natural	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
\hbar	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
Δ	\natural	\hbar	Δ	$\underline{\vee}$	\triangleright	$\underline{\vee}$
\wp	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
\triangleright	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$
$\underline{\vee}$	\natural	\hbar	Δ	\wp	\triangleright	$\underline{\vee}$

Clearly $\{\mathfrak{h}\} \subseteq \{\mathfrak{h}, \mathfrak{h}\} \subseteq \{\mathfrak{h}, \mathfrak{h}, \Delta\}$ and $\{\mathfrak{h}, \mathfrak{h}, \Delta\} \alpha_2 \{\mathfrak{h}, \mathfrak{h}, \wp\} = \{\mathfrak{h}, \mathfrak{h}\}$. But $\{\mathfrak{h}, \mathfrak{h}, \Delta\} \not\subseteq \{\mathfrak{h}, \mathfrak{h}\}$ and $\{\mathfrak{h}, \mathfrak{h}, \wp\} \not\subseteq \{\mathfrak{h}, \mathfrak{h}\}$. Hence, $\{\mathfrak{h}, \mathfrak{h}\}$ is a T^1 -KId but not a T^1 -PKId.

Theorem 3.9. *If \mathfrak{R} is an T^1 -KId of \mathfrak{Z} and \mathfrak{D} is a τ_2^2 -system of \mathfrak{Z} , then $\mathfrak{R} \cap \mathfrak{D} = \phi$, then there exists a T^1 -PKId \mathfrak{P} of \mathfrak{Z} , then $\mathfrak{P} \cap \mathfrak{D} = \phi$ with $\mathfrak{R} \subseteq \mathfrak{P}$.*

Proof. Let \mathfrak{R} be a T^1 -KId in \mathfrak{Z} and let \mathfrak{D} be a τ_2^2 -system in \mathfrak{Z} , then $\mathfrak{R} \cap \mathfrak{D} = \phi$. Now the set $\mathfrak{J} = \{I \mid I \text{ is an } T^1\text{-KId } \mathfrak{R} \subseteq \mathfrak{J} \text{ and } \mathfrak{J} \cap \mathfrak{D} = \phi\}$. Let $\mathfrak{R}_1 \subseteq \mathfrak{R}_2 \subseteq \mathfrak{R}_3 \dots$ be any chain in \mathfrak{J} . Let $\mathfrak{R} = \cup \mathfrak{R}_i$. Clearly \mathfrak{R} is a T^1 -KId in \mathfrak{Z} and $\mathfrak{R} \cap \mathfrak{D} = \phi$. By Zorn's lemma, \mathfrak{J} has contains maximal element \mathfrak{P} . Assume there are T^1 -KIds \mathfrak{U} and \mathfrak{V} in \mathfrak{Z} , then $\mathfrak{U} \alpha_2 \mathfrak{V} \subseteq \mathfrak{P}$ but $\mathfrak{U} \not\subseteq \mathfrak{P}$ and $\mathfrak{V} \not\subseteq \mathfrak{P}$, then $\varrho \in \mathfrak{U} \setminus \mathfrak{P}$ and $\varrho \in \mathfrak{V} \setminus \mathfrak{P}$. Hence, $\mathfrak{P} \subset \mathfrak{P} \cup \overline{(\varrho)}$ and $\overline{\mathfrak{P}} \subset \mathfrak{P} \cup \overline{(\varrho)}$. By maximal property, $\mathfrak{P} \tau_1^2 \in (\mathfrak{P} \cup \overline{(\varrho)}) \cap \mathfrak{D}$ and $\tau_2^2 \in (\mathfrak{P} \cup \overline{(\varrho)}) \cap \mathfrak{D}$. Then there exists $\tau_1^2 \in \overline{(\tau_1^2)}$ and $\tau_2^2 \in \overline{(\tau_2^2)}$, then $\tau_1^2 \alpha_2 \tau_2^2 \in \mathfrak{D}$. Now

$$\begin{aligned} (\mathfrak{P} \cup \overline{(\varrho)}) \alpha_2 \mathfrak{P} \cup \overline{(\varrho)} &= \mathfrak{P} \cup (\mathfrak{P} \alpha_2 \overline{(\varrho)}) \cup (\overline{(\varrho)} \alpha_2 \mathfrak{P}) \cup (\overline{(\varrho)} \alpha_2 \overline{(\varrho)}) \\ &\subseteq \mathfrak{P} \cup \mathfrak{P} \cup \mathfrak{P} \cup (\mathfrak{U} \alpha_2 \mathfrak{V}) \\ &\subseteq \mathfrak{P} \end{aligned}$$

By Lemma 2.6 $\overline{(\tau_1^2)} \alpha_2 \overline{(\tau_2^2)} \subseteq \overline{(\tau_1^2)} \alpha_2 \overline{(\tau_2^2)} \subseteq \mathfrak{P}$. Hence, $\mathfrak{P} \cap \mathfrak{D} \neq \phi$, which is contradiction to $\mathfrak{P} \cap \mathfrak{D} = \phi$. Hence, \mathfrak{P} is a T^1 -PKId of \mathfrak{Z} .

Corollary 3.10. *If \mathfrak{R} is an T^1 -KId of \mathfrak{Z} and \mathfrak{D} is a τ_0^2 -system (τ_1^2 -system) of \mathfrak{Z} , then $\mathfrak{R} \cap \mathfrak{D} = \phi$, then there exists a T^1 -PKId \mathfrak{P} of \mathfrak{Z} , then $\mathfrak{P} \cap \mathfrak{D} = \phi$ with $\mathfrak{R} \subseteq \mathfrak{P}$.*

4. Classification of Type-2 prime k-ideals and m-systems of bi-semirings

Definition 4.1. (i) *The KT^2 -Id \mathfrak{P} is represent a PKT^2 -Id if $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$ for the KT^2 -Ids \mathfrak{R} and \mathfrak{J} in \mathfrak{Z}*

(ii) *The KT^2 -Id \mathfrak{P} is represent a 0- PKT^2 -Id if $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$ for the T^2 -Ids \mathfrak{R} and \mathfrak{J} in \mathfrak{Z}*

(iii) *The KT^2 -Id \mathfrak{P} is represent a 1- PKT^2 -Id if $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P} \Leftrightarrow \mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq P$ for the KT^2 -Ids \mathfrak{R} and T^2 -Id \mathfrak{J} in \mathfrak{Z} .*

Theorem 4.2. *If \mathfrak{P} is a T^2 -KId in \mathfrak{Z} , then following conditions are equivalent.*

- (i) \mathfrak{P} is T^2 -PKId.
- (ii) $\epsilon \alpha_3 \mathfrak{Z} \alpha_3 \gamma \subseteq \mathfrak{P}$ implies either $\epsilon \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (iii) $(\epsilon)_{tri} \alpha_3 (\gamma)_{tri} \subseteq \mathfrak{P}$ implies either $\epsilon \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (iv) For the T^2 -TRIDs \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (v) $(\epsilon) \alpha_3 (\gamma) \subseteq \mathfrak{P}$ implies either $\epsilon \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.
- (vi) If the T^2 -Ids \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (vii) If the T^2 -KIds \mathfrak{R} and Ids \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (viii) If the T^2 -Ids \mathfrak{R} and KIds \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (ix) If the T^2 -RIDs \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (x) If the T^2 -RKIDs \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xi) If the T^2 -RKIDs \mathfrak{R} and T^2 -RIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xii) If the T^2 -RIDs \mathfrak{R} and T^2 -RKIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xiii) If the T^2 -LIDs \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xiv) If the T^2 -LKIDs \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xv) If the T^2 -LKIDs \mathfrak{R} and T^2 -LIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xvi) If the T^2 -LIDs \mathfrak{R} and T^2 -LKIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xvii) If the T^2 -RIDs \mathfrak{R} and T^2 -LIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xviii) If the T^2 -RKIDs \mathfrak{R} and T^2 -LKIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xix) If the T^2 -RIDs \mathfrak{R} and T^2 -LKIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.
- (xx) If the T^2 -RKIDs \mathfrak{R} and T^2 -LIDs \mathfrak{J} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ implies either $\mathfrak{R} \subseteq \mathfrak{P}$ or $\mathfrak{J} \subseteq \mathfrak{P}$.

Proof. (i) \Rightarrow (ii) : Let \mathfrak{P} be a PWKId. If $\epsilon \alpha_3 \mathfrak{Z} \alpha_3 \gamma \subseteq \mathfrak{P}$, then $\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z} \alpha_3 \mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z} \subseteq \mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z} \subseteq \mathfrak{Z} \alpha_3 \mathfrak{P} \alpha_3 \mathfrak{Z} \subseteq \mathfrak{P}$.

By Lemma 2.6 $\overline{\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z} \alpha_3 \mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z}} \subseteq \mathfrak{P}$. Hence, $\overline{\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z}} \subseteq \mathfrak{P}$ or $\overline{\mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z}} \subseteq \mathfrak{P}$. Assume $\overline{\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z}} \subseteq \mathfrak{P}$, then $(\epsilon) \alpha_3 (\epsilon) \alpha_3 (\epsilon) \subseteq \overline{\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z}} \hookrightarrow (\epsilon) \subseteq \mathfrak{P}$. Hence, $\epsilon \in \mathfrak{P}$. Similarly $\overline{\mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z}} \subseteq \mathfrak{P} \hookrightarrow \gamma \in \mathfrak{P}$.

(ii) \Rightarrow (v) : If $(\epsilon) \alpha_3 (\gamma) \subseteq \mathfrak{P}$, then $\epsilon \alpha_3 \mathfrak{Z} \alpha_3 \gamma \subseteq (\epsilon) \alpha_3 (\gamma) \subseteq \mathfrak{P}$. By (ii) $\epsilon \in \mathfrak{P}$ or $\gamma \in \mathfrak{P}$.

(v) \Rightarrow (vi) : Let \mathfrak{R} and \mathfrak{J} be the T^2 -Ids in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$. If $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ $(\epsilon) \alpha_3 (\gamma) \subseteq \mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P} \hookrightarrow \gamma \in \mathfrak{P}$. Hence, $\mathfrak{J} \subseteq \mathfrak{P}$.

(vi) \Rightarrow (vii)(vii) \Rightarrow (i)(vi) \Rightarrow (viii)(xii) \Rightarrow (x) and (i) \hookrightarrow (ii) \hookrightarrow (iii) \hookrightarrow (iv) \hookrightarrow (vi) are direct.

(viii) \Rightarrow (xii) : If the T^2 -RId \mathfrak{R} and T^2 -RKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon) \alpha_3 (\gamma) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \subseteq \mathfrak{P}$. By Lemma 3.2 and Lemma 2.6 $(\epsilon) \alpha_3 (\overline{\gamma}) \subseteq (\epsilon) \alpha_2 (\overline{\gamma}) \subseteq \mathfrak{P}$. Then by (viii) $\overline{\gamma} \in \mathfrak{P} \forall \gamma \in \mathfrak{J}$. Hence, $\mathfrak{J} \subseteq \mathfrak{P}$.

(vi) \Rightarrow (ix) : Let \mathfrak{R} and \mathfrak{J} be the T^2 -RId of \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$. If $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\delta \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\chi \in \mathfrak{R}$ and by Lemma 3.2 $(\chi) \alpha_3 (\delta) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \subseteq \mathfrak{P}$. Then by (vi) $\chi \in \mathfrak{P}$. Hence, $\mathfrak{R} \subseteq \mathfrak{P}$.

(ix) \Rightarrow (x) and (xi) \Rightarrow (x) are direct.

(x) \Rightarrow (i) : If the T^2 -RKIds \mathfrak{R} and \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon)_r \alpha_3 (\gamma)_r \subseteq \mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$.

By Lemma 2.6 $(\epsilon)_r \alpha_3 (\gamma)_r \subseteq (\epsilon)_r \alpha_3 (\gamma)_r \subseteq \mathfrak{P}$. Hence, by (x) $\mathfrak{J} \subseteq \mathfrak{P}$.

(vii) \Rightarrow (xi) : If the T^2 -RKId \mathfrak{R} and T^2 -RId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\omega \in \mathfrak{R} \setminus \mathfrak{P}$ and $\nu \in \mathfrak{J} \setminus \mathfrak{P}$. Then $(\omega) \alpha_3 (\nu) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \subseteq \mathfrak{P}$ and by Lemma 2.6 $(\omega) \alpha_3 (\nu) \subseteq (\omega) \alpha_3 (\nu) \subseteq \mathfrak{P}$, which contradicts (vii). Hence, (xi) holds.

(viii) \Rightarrow (xvi) : If the T^2 -LId \mathfrak{R} and T^2 -LKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\gamma \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\epsilon \in \mathfrak{R}$ and by Lemma 3.2 $(\epsilon) \alpha_3 (\gamma) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \subseteq \mathfrak{P}$. By Lemma 2.6 $(\epsilon) \cdot (\overline{\gamma}) \subseteq (\epsilon) \alpha_3 (\overline{\gamma}) \subseteq \mathfrak{P}$. Hence, by (viii) $\mathfrak{R} \subseteq \mathfrak{P}$.

(xvi) \Rightarrow (xiv)(xiv) \Rightarrow (i)(xiii) \Rightarrow (xiv) are direct.

(vi) \Rightarrow (xiii) : If \mathfrak{R} and \mathfrak{J} are the T^2 -LIds of \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{R} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\chi \in \mathfrak{R} \setminus \mathfrak{P}$ and $\delta \in \mathfrak{J} \setminus \mathfrak{P}$. By Lemma 3.2 $(\chi) \alpha_3 (\delta) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \subseteq \mathfrak{P}$, which contradicts (vi). Hence, (xiii) holds.

(vii) \Rightarrow (xv) : If the T^2 -LKId \mathfrak{R} and T^2 -LId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\nu \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\omega \in \mathfrak{R}$ and by Lemma 3.2 $(\omega) \alpha_3 (\nu) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \subseteq \mathfrak{P}$. By Lemma 2.6 $(\omega) \alpha_3 (\nu) \subseteq (\omega) \alpha_3 (\nu) \subseteq \mathfrak{P}$. Hence, by (vii) $\mathfrak{R} \subseteq \mathfrak{P}$.

(xv) \Rightarrow (xiv)(xvii) \Rightarrow (xx)(xx) \Rightarrow (xviii) and (xviii) \Rightarrow (i) are direct.

(vi) \Rightarrow (xvii) : If the T^2 -RId \mathfrak{R} and T^2 -LId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ with $\mathfrak{R} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{R} \setminus \mathfrak{P}$. For any $\gamma \in \mathfrak{J}$ and by Lemma 3.2 $(\epsilon) \alpha_3 (\gamma) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \subseteq \mathfrak{P}$.

$$\begin{aligned} (\epsilon) \alpha_3 (\gamma) &= ((n \alpha_3 \epsilon) \cup (\epsilon \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \epsilon) \cup (\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z})) \alpha_3 ((m \alpha_3 \gamma) \cup (b \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \gamma) \cup (\mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z})) \\ &\subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \\ &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{P}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{P} \alpha_3 \mathfrak{Z}) \\ &\subseteq \mathfrak{P}. \end{aligned}$$

Hence, by (vi) $\mathfrak{J} \subseteq \mathfrak{P}$.

(vi) \Rightarrow (xix) : If the T^2 -RId \mathfrak{R} and T^2 -LKId \mathfrak{J} in \mathfrak{Z} , then $\mathfrak{R} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\vartheta \in \mathfrak{J} \setminus \mathfrak{P}$. For any $\rho \in \mathfrak{R}$ and by Lemma 3.2 $(\rho) \alpha_3 (\vartheta) \subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \subseteq \mathfrak{P}$. Now

$$\begin{aligned} (\epsilon) \alpha_3 (\gamma) &= ((n \alpha_3 \epsilon) \cup (\epsilon \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \epsilon) \cup (\mathfrak{Z} \alpha_3 \epsilon \alpha_3 \mathfrak{Z})) \alpha_3 ((m \alpha_3 \gamma) \cup (\gamma \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \gamma) \cup (\mathfrak{Z} \alpha_3 \gamma \alpha_3 \mathfrak{Z})) \\ &\subseteq (\mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{R} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{Z}) \\ &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_3 \mathfrak{Z}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{P}) \cup (\mathfrak{Z} \alpha_3 \mathfrak{P} \alpha_3 \mathfrak{Z}) \\ &\subseteq \mathfrak{P}. \end{aligned}$$

Now

$$\begin{aligned}
 (\varrho) \alpha_3 (\vartheta) &= ((n \alpha_3 \varrho) \cup (\varrho \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \varrho) \cup (\mathfrak{J} \alpha_3 \varrho \alpha_3 \mathfrak{J})) \alpha_3 ((m \alpha_3 \vartheta) \cup (\vartheta \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \vartheta) \cup (\mathfrak{J} \alpha_3 \vartheta \alpha_3 \mathfrak{J})) \\
 &\subseteq (\mathfrak{K} \alpha_3 \mathfrak{J}) \cup (\mathfrak{K} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \mathfrak{K} \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \mathfrak{K} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{J}) \\
 &\subseteq \mathfrak{P} \cup (\mathfrak{P} \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \mathfrak{P}) \cup (\mathfrak{J} \alpha_3 \mathfrak{P} \alpha_3 \mathfrak{J}) \\
 &\subseteq \mathfrak{P}.
 \end{aligned}$$

Hence, by (vi) $\mathfrak{K} \subseteq \mathfrak{P}$.

(xix) \Rightarrow (i): is direct.

(vi) \Rightarrow (xx) : If the T^2 -RKId \mathfrak{K} and T^2 -LId \mathfrak{J} in \mathfrak{J} , then $\mathfrak{K} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{K} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{K} \setminus \mathfrak{P}$ and $\gamma \in \mathfrak{J} \setminus \mathfrak{P}$. By Lemma 3.2 $(\epsilon) \alpha_3 (\gamma) \subseteq (\mathfrak{K} \alpha_3 \mathfrak{J}) \cup (\mathfrak{K} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \mathfrak{K} \alpha_3 \mathfrak{J}) \cup (\mathfrak{J} \alpha_3 \mathfrak{K} \alpha_3 \mathfrak{J} \alpha_3 \mathfrak{J}) \subseteq \mathfrak{P}$, which contradicts (vi). Hence, (xx) holds.

Definition 4.3. The subset \mathfrak{D} of \mathfrak{J} is represent

- (i) τ_0^3 -system if $\zeta, \eta \in \mathfrak{D}$ and exists $s \in \mathfrak{J} \hookrightarrow \zeta \alpha_3 s \alpha_3 \eta \in \mathfrak{D}$.
- (ii) τ_1^3 -system if $\zeta, \eta \in \mathfrak{D}$ and exists $\zeta \in (\underline{\zeta}), \eta \in (\underline{\eta}) \hookrightarrow \zeta \alpha_3 \eta \in \mathfrak{D}$.
- (iii) τ_2^3 -system if $\zeta, \eta \in \mathfrak{J}$ and exists $\zeta \in (\underline{\zeta}), \eta \in (\underline{\eta}) \hookrightarrow \zeta \alpha_3 \eta \in \mathfrak{D}$.

Lemma 4.4. The T^2 -KId \mathfrak{P} is a T^2 -PKId if and only if \mathfrak{P}^c is a $\tau_0^3(\tau_1^3\tau_2^3)$ -system.

Proof. Let \mathfrak{P} be an T^2 -PKId in \mathfrak{J} and $\epsilon, \gamma \in \mathfrak{P}^c$. Then by Theorem 4.2 $\epsilon \alpha_3 \mathfrak{J} \alpha_3 \gamma \not\subseteq \mathfrak{P}$. Then there exists $\chi \in \mathfrak{J}$, then $\epsilon \alpha_3 \chi \alpha_3 \gamma \in \mathfrak{P}^c$. Hence, \mathfrak{P}^c is a τ_0^3 -system. For $\epsilon \alpha_3 \gamma \in \mathfrak{P}^c$ by Theorem 4.2 (vii) and (x) $(\epsilon) \alpha_3 (\gamma) \not\subseteq \mathfrak{P}$ and $(\epsilon) \alpha_3 (\underline{\gamma}) \not\subseteq \mathfrak{P}$. Then \mathfrak{P}^c is a τ_1^3 -system and τ_2^3 -system respectively.

Conversely if there exist T^2 -KIds \mathfrak{K} and \mathfrak{J} , then $\mathfrak{K} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$ but $\mathfrak{K} \not\subseteq \mathfrak{P}$ and $\mathfrak{J} \not\subseteq \mathfrak{P}$, then $\epsilon \in \mathfrak{K} \setminus \mathfrak{P}$ and $g \in \mathfrak{J} \setminus \mathfrak{P}$. Hence, $\epsilon \alpha_3 g \in \mathfrak{P}^c \hookrightarrow \epsilon_1 \alpha_3 \gamma_1 \in \mathfrak{P}^c$ for some $\epsilon_1 \in (\underline{\epsilon})$ and $\gamma_1 \in (\underline{\gamma})$, which contradicts $\mathfrak{K} \alpha_3 \mathfrak{J} \subseteq \mathfrak{P}$.

Remark 4.5. The Example 4.6 asserts that there are T^2 -KIds, which are not T^2 -PKIds.

Example 4.6. Let $\mathfrak{J} = \{\natural, \hbar, \Delta, \wp, \blacktriangleright, \blacktriangledown\}$ be the bi-semiring.

α_1	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\natural	\natural	\natural	\natural	\wp	\wp	\blacktriangledown
\hbar	\natural	\hbar	\hbar	\wp	\wp	\blacktriangledown
Δ	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\wp	\natural	\wp	\wp	\wp	\wp	\blacktriangledown
\blacktriangleright	\natural	\wp	\blacktriangleright	\wp	\blacktriangleright	\blacktriangledown
\blacktriangledown	\natural	\wp	\blacktriangledown	\wp	\blacktriangledown	\blacktriangledown

α_2	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\natural	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\hbar	\natural	\hbar	Δ	\hbar	\blacktriangleright	\blacktriangledown
Δ	\natural	\hbar	Δ	\hbar	\blacktriangleright	\blacktriangledown
\wp	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\blacktriangleright	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\blacktriangledown	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown

α_3	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\natural	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\hbar	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
Δ	\natural	\hbar	Δ	\blacktriangledown	\blacktriangleright	\blacktriangledown
\wp	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\blacktriangleright	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown
\blacktriangledown	\natural	\hbar	Δ	\wp	\blacktriangleright	\blacktriangledown

Now $\{\natural, \blacktriangledown\} \alpha_3 \{\natural, \Delta\} = \{\natural\} \subseteq \{\natural, \hbar\}$. Hence, $\{\natural, \hbar\}$ is T^2 -KId but not a T^2 -PKId.

Theorem 4.7. If \mathfrak{K} is an T^2 -KId of \mathfrak{J} and \mathfrak{D} is a τ_2^3 -system of \mathfrak{J} , then $\mathfrak{K} \cap \mathfrak{D} = \phi$, then there exists a T^2 -PKId \mathfrak{P} of \mathfrak{J} , then $\mathfrak{P} \cap \mathfrak{D} = \phi$ with $\mathfrak{K} \subseteq \mathfrak{P}$.

Proof. Let R be a T^2 -KId in \mathfrak{Z} and let \mathfrak{J} be a τ_2^3 -system in \mathfrak{Z} , then $\mathfrak{R} \cap \mathfrak{D} = \phi$. Now we consider the set $\mathfrak{J} = \{I \mid I \text{ is an } T^2\text{-KId } \mathfrak{R} \subseteq \mathfrak{J} \text{ and } \mathfrak{J} \cap \mathfrak{D} = \phi\}$. Clearly \mathfrak{J} is non-empty. Let $\mathfrak{R}_1 \subseteq \mathfrak{R}_2 \subseteq \mathfrak{R}_3$ be any chain in \mathfrak{J} . Let $\mathfrak{N} = \cup \mathfrak{R}_i$.

Clearly \mathfrak{N} is a T^2 -KId in \mathfrak{Z} and $\mathfrak{N} \cap \mathfrak{D} = \phi$.

By Zorn's lemma \mathfrak{J} has contains maximal element \mathfrak{P} . Assume there are T^2 -KIds \mathcal{U} and \mathcal{V} in \mathfrak{Z} , then $\mathcal{U} \alpha_3 \mathcal{V} \subseteq \mathfrak{P}$ but $\mathcal{U} \not\subseteq \mathfrak{P}$ and $\mathcal{V} \not\subseteq \mathfrak{P}$, then $\varrho \in \mathcal{U} \setminus \mathfrak{P}$ and $\varrho \in \mathcal{V} \setminus \mathfrak{P}$. Hence, $\mathfrak{P} \subset \mathfrak{P} \cup \overline{(\varrho)}$ and $\mathfrak{P} \subset \mathfrak{P} \cup \overline{(\mathcal{G})}$. By Maximal property \mathfrak{P} , $\tau_2^3 \in (\mathfrak{P} \cup \overline{(\varrho)}) \cap \mathfrak{D}$ and $\tau_2^3 \in (\mathfrak{P} \cup \overline{(\mathcal{G})}) \cap \mathfrak{D}$. Then there exists $\tau_1^{3'} \in (\tau_1^3)$ and $\tau_2^{3'} \in (\tau_2^3)$, then $\tau_1^{3'} \alpha_3 \tau_2^{3'} \in \mathfrak{D}$. Now

$$\begin{aligned} \mathfrak{P} \cup \overline{(\varrho)} \alpha_3 \mathfrak{P} \cup \overline{(\mathcal{G})} &= \mathfrak{P} \cup (\mathfrak{P} \alpha_3 \overline{(\mathcal{G})}) \cup (\overline{(\varrho)} \alpha_3 \mathfrak{P}) \cup (\overline{(\varrho)} \alpha_3 \overline{(\mathcal{G})}) \\ &\subseteq \mathfrak{P} \cup \mathfrak{P} \cup \mathfrak{P} \cup (\mathcal{U} \alpha_3 \mathcal{V}) \\ &\subseteq \mathfrak{P} \end{aligned}$$

By Lemma 2.6 $\overline{(\tau_1^3)} \alpha_3 \overline{(\tau_2^3)} \subseteq \overline{(\tau_1)} \alpha_3 \overline{(\tau_2)} \subseteq \mathfrak{P}$. Hence, $\mathfrak{P} \cap \mathfrak{D} \neq \phi$, which is contradiction, $\mathfrak{P} \cap \mathfrak{D} = \phi$. Hence, \mathfrak{P} is a T^2 -PKId of \mathfrak{Z} .

Corollary 4.8. *If \mathfrak{R} is an T^2 -KId of \mathfrak{Z} and \mathfrak{D} is a τ_0^3 -system (τ_1^3 -system) of \mathfrak{Z} , then $\mathfrak{R} \cap \mathfrak{D} = \phi$, then there exists a T^2 -PKId \mathfrak{P} of \mathfrak{Z} , then $\mathfrak{P} \cap \mathfrak{D} = \phi$ with $\mathfrak{R} \subseteq \mathfrak{P}$.*

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